Math 405/538 Differential Geometry
Final Exam

January 7, 2013

1a) (3 pts) Define torsion of a regular curve in $\mathbb{R}^3$.
1b) (3 pts) Define Gaussian Curvature of a surface in $\mathbb{R}^3$.
1c) (3 pts) State the Fundamental Theorem of Curves.
1d) (3 pts) State the Fundamental Theorem of Surfaces.
1e) (3 pts) State the Gauss-Bonnet Theorem.

2) (3 pts each) TRUE-FALSE.

a. For any given smooth functions $\kappa(s) > 0$ and $\tau(s)$, there exists a regular curve $\alpha(s)$ in $\mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$.

b. For any given smooth functions E, F, G and e, f, g, there exists a regular surface S with $I_S = Ex^2+2Fxy+Gy^2$ and $II_S = ex^2+2fy^2+gy^2$.

c. Let L be a straight line in a surface S in $\mathbf{R}^3$. Then, for any $p \in L$ the direction given by L is principal direction.

d. Let S be a disk in $\mathbf{R}^3$ with Gaussian curvature $K \leq 0$ everywhere. Then, any two different geodesics starting at the same point cannot meet again in S.

e. Let S_1 and S_2 are two surfaces which are tangent to each other along a curve α. If α is a geodesic of S_1, then it is a geodesic of S_2, too.

3a) (10 pts) Calculate the Frenet apparatus $(\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau)$ for

$$\alpha(t) = (\cos t, \sin t, t) \quad t \in (-1, 1)$$

3b. (6 pts) Let S be the cylinder $\{x^2 + y^2 = 1\}$ in $\mathbb{R}^3$. Then, the curve above $\alpha \subset S$. Compute the normal curvature $\kappa_n(t)$ and geodesic curvature $\kappa_g(t)$ of α.

4) Let S be a surface of revolution with parametrization

$$\varphi(u, v) = (\phi(u)\cos v, \phi(u)\sin v, u)$$

4a. (8 pts) Find the principal directions, and compute the Gaussian curvature $K(u, v)$ for any point $p = \varphi(u, v)$.
4b. (8 pts) Show that any meridian curve $(v = v_o)$ is a geodesic.
Show that a parallel curve $(u = u_o)$ is a geodesic if and only if $\varphi'(u_o) = 0$.

5) For $c > 0$, define the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $F(p) = 3p$.
5a. (8 pts) Let $\alpha(s)$ be a regular curve in $\mathbb{R}^3$, and let $\widehat{\alpha}(s) = F(s)$. Compute $\widehat{\kappa}(s)$ and $\widehat{\tau}(s)$ in terms of $\kappa(s)$ and $\tau(s)$.

5b. (8 pts) Let S be a regular surface, and let $\widehat{S} = F(S)$. Compute Gaussian curvature $\widehat{K}(p)$ in terms of $K(p)$.

6a) (10 pts) Let S be a smooth, closed, orientable surface in $\mathbb{R}^3$. Show that the Gauss Map $N : S \to S^2$ is surjective.
6b) (8 pts) Let $K_+(p) = \max\{0, K(p)\}$. Show that $\int_S K_+ \, dA \geq 4\pi$.

7) (24 pts) Show that if all geodesics of a connected surface S are plane curves, then S is contained in a plane or a sphere.

[Hint: You can use the following steps.]

Step 1. If a geodesic is a plane curve, then it is a line of curvature.
Step 2. If all geodesics of a connected surface S are plane curves, then every point of S is an umbilical point (principal curvatures are same, $k_1 = k_2$).
Step 3. If every point of S is an umbilical point, then S is contained in a plane or a sphere.

KEY.

2) (3 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

a. For any given smooth functions $\kappa(s) > 0$ and $\tau(s)$, there exists a regular curve $\alpha(s)$ in $\mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$.

TRUE. Fund. Thm. of Curves. - existence.

b. For any given smooth functions E, F, G and e, f, g, there exists a regular surface S with $I_S = Ex^2+2Fxy+Gy^2$ and $II_S = ex^2+2fy^2+gy^2$.

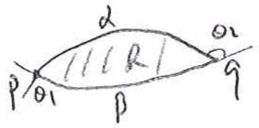
FALSE. Fund Thm of Surfaces - Gauss + Codazzi Eqns.

c. Let L be a straight line in a surface S in R^3. Then, for any $p \in L$ the direction given by L is principal direction.

FALSE. $S: \{x^2+y^2-z^2=1\}$ is a ruled surface.

d. Let S be a ~~surface~~ disk in R^3 with Gaussian curvature $K \leq 0$ everywhere. Then, any two different geodesics starting at the same point cannot meet again in S.

TRUE.



$\int_R K\,dA + \theta_1+\theta_2 = 2\pi$

$\int_R K\,dA \leq 0$, $\theta_1 < \pi$, $\theta_2 < \pi$ ✗.

e. Let S_1 and S_2 are two surfaces which are tangent to each other along a curve α. If α is a geodesic of S_1, then it is a geodesic of S_2, too.

TRUE. $k_n^1 \,//\, N \Rightarrow k_n^2 \,//\, N$ ✓

$\vec{T}' = \quad k\vec{N}$
$\vec{N}' = -k\vec{T} \quad - \tau\vec{B}$
$\vec{B}' = \quad \tau\vec{N}$

3a) (10 pts) Calculate the Frenet apparatus $(\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau)$ for

$$\alpha(t) = (\cos t, \sin t, t) \quad t \in (-1, 1)$$

Parametrize by arclength: $\alpha(s) = \langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \rangle$

$\vec{T} = \alpha'(s) = \frac{1}{\sqrt{2}}\langle -\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1 \rangle$ | $\alpha''(s) = \frac{1}{2}\langle -\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0 \rangle$

$\boxed{k = |\alpha''(s)| = \frac{1}{2}}$ $\quad \vec{N} = \langle -\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0 \rangle$

$$\vec{B} = \begin{vmatrix} i & j & k \\ -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} & -\sin\frac{s}{\sqrt{2}} & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\langle +\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 1 \rangle$$

$\vec{B}' = \frac{1}{\sqrt{2}}\langle \cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0 \rangle$

$B' = \frac{1}{\sqrt{2}} N \Rightarrow \boxed{\tau = \frac{1}{\sqrt{2}}}$

3b. (6 pts) Let S be the cylinder $\{x^2 + y^2 = 1\}$ in $\mathbb{R}^3$. Then, the curve above $\alpha \subset S$. Compute the normal curvature $\kappa_n(t)$ and geodesic curvature $\kappa_g(t)$ of α.

$S: \varphi(u,v) = \langle \cos u, \sin u, v \rangle \Rightarrow \vec{N} = \varphi_u \times \varphi_v = \langle \cos u, \sin u, 0 \rangle$

$\Rightarrow \alpha'' \parallel \vec{N} \Rightarrow k = k_n = \frac{1}{2}$ and $k_g = 0$ 0

4) Let S be a surface of revolution with parametrization

$$\varphi(u,v) = (\phi(u)\cos v, \phi(u)\sin v, u)$$

4a. (8 pts) Find the principal directions, and compute the Gaussian curvature $K(u,v)$ for any point $p = \varphi(u,v)$.

$\varphi_u = \langle \phi'(u)\cos v, \phi'(u)\sin v, 1\rangle$ $\quad |\varphi_u| = \sqrt{1+\phi'(u)^2}$ $\quad E = 1+\phi'(u)^2$

$\varphi_v = \langle -\phi(u)\sin v, -\phi(u)\cos v, 0\rangle$ $\quad |\varphi_v| = |\phi(u)|$ $\quad \Rightarrow F = 0$, $G = \phi(u)^2$

$\varphi_u \times \varphi_v = \langle \phi(u)\cos v, \phi(u)\sin v, -\phi'(u)\phi(u)\rangle \Rightarrow N = \dfrac{\phi(u)}{\sqrt{1+\phi'^2}}\langle \cos v, \sin v, -\phi'\rangle$

$\varphi_{uu} = \langle \phi''\cos v, \phi''\sin v, 0\rangle$

$\varphi_{uv} = \langle -\phi'\sin v, \phi'\cos v, 0\rangle \Rightarrow$ $e = \dfrac{\phi\cdot\phi''}{\sqrt{1+\phi'^2}}$, $f = 0$, $g = \dfrac{\phi^2}{\sqrt{1+\phi'^2}}$

$\varphi_{vv} = \langle -\phi\cos v, -\phi\sin v, 0\rangle$

$S_p = \begin{bmatrix} E & F \\ F & G\end{bmatrix}^{-1}\begin{bmatrix} e & f \\ f & g\end{bmatrix}$

$\Rightarrow$ φ_u and φ_v are eigenvectors

$k_1 = \dots \quad k_2 = \dots$

$K = \dfrac{eg - f^2}{EG - F^2} = \phi\cdot\phi''$

4b. (8 pts) Show that any meridian curve $(v = v_o)$ is a geodesic.
Show that a parallel curve $(u = u_o)$ is a geodesic if and only if $\varphi'(u_o) = 0$.

α geodesic $\Leftrightarrow$ $N_\alpha \,//\, N_S$.

$\alpha_u = \langle \phi(u)\cos v_o, \phi(u)\sin v_o, u\rangle$ is a plane curve.

$+\ T_\alpha \perp N_\alpha \Rightarrow N_\alpha \,//\, N_S$

$\alpha_v = \langle \phi(u_0)\cos v, \phi(u_0)\sin v, u_0\rangle \Rightarrow N_{\alpha_v} = \langle -\cos v, -\sin v, 0\rangle$

$N_{\alpha v} = N_S = \langle \cos v, \sin v, -\phi'\rangle \Leftrightarrow \phi'(u_0) = 0.$

5) For $c > 0$, define the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $F(p) = 3p$.

5a. (8 pts) Let $\alpha(s)$ be a regular curve in $\mathbb{R}^3$, and let $\widehat{\alpha}(s) = F(s)$. Compute $\widehat{\kappa}(s)$ and $\widehat{\tau}(s)$ in terms of $\kappa(s)$ and $\tau(s)$.

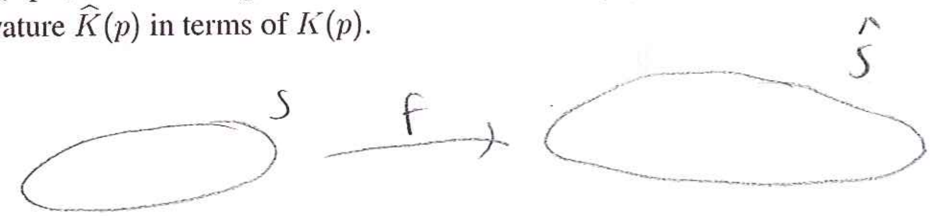
$\hat\alpha(s) = 3\alpha(\frac{s}{3}) \quad \Rightarrow \quad \hat\alpha' = \alpha'(\frac{s}{3}) \quad \hat\alpha''(s) = \frac{1}{3}\alpha''(\frac{s}{3})$

par. by arclength

$\Rightarrow \quad \hat\kappa(s) = \frac{1}{3}\kappa(\frac{s}{3})$

Sim. $\quad \hat\tau(s) = \frac{1}{3}\tau(\frac{s}{3})$

5b. (8 pts) Let S be a regular surface, and let $\widehat{S} = F(S)$. Compute Gaussian curvature $\widehat{K}(p)$ in terms of $K(p)$.



$\alpha \subseteq S \Rightarrow F\circ\alpha = \hat\alpha \subseteq \hat S \qquad \hat k_{\hat\alpha} = \frac{k_\alpha}{3}$

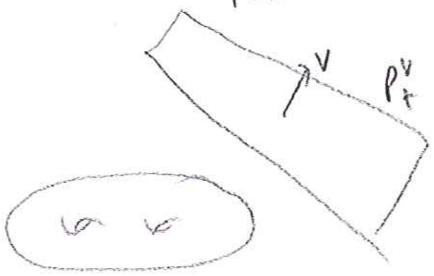
For any p, let k_1 and k_2 principal curvatures with α_1 and α_2 curves in principal directions.

By above $\hat\alpha_1$ and $\hat\alpha_2$ are principal directions at $\hat p = F(p)$

$\Rightarrow \quad \hat K_{\hat p} = \hat k_1 \cdot \hat k_2 = \frac{k_1}{3}\cdot\frac{k_2}{3} = \frac{K}{9} \quad \Rightarrow \quad \boxed{\hat K_{\hat p} = \frac{K_p}{9}}$

6a) (10 pts) Let S be a smooth, closed, orientable surface in $\mathbb{R}^3$. Show that the Gauss Map $N : S \to S^2$ is surjective.

Fix a unit vector $v \in S^2$. let P_t^v is the plane $v \cdot p = t$



For t very large, $P_t^v \cap S = \emptyset$

$t \searrow$, Find the first point of contact. at $p_0 \in S$.

$\Rightarrow \quad T_{p_0} S = P_{t_0}^v \Rightarrow N(p_0) = v \quad \square$.

6b) (8 pts) Let $K_+(p) = \max\{0, K(p)\}$. Show that $\int_S K_+ \, dA \geq 4\pi$.

$$\int_S K_+ dA = \int_{S^+} K dA \quad \text{where} \quad S^+ = \{ p \in S \mid K(p) \geq 0 \}$$

$$\int_{S^+} K dA = \int_{S^+} \det(dN) dA = \overset{\text{area}}{|N(S^+)|}.$$

Claim: $N : S^+ \to S^2$ surjective

By above proof, at the first point of contact p_0, S lies in one side of the tangent plane $\Rightarrow K(p_0) \geq 0$. $\Rightarrow N : S^+ \to S^2$ onto

$\Rightarrow \quad |N(S^+)| \geq |S^2| = 4\pi$.

Bonus) (20 pts) Show that if all geodesics of a connected surface S are plane curves, then S is contained in a plane or a sphere.

[Hint: You can use the following steps.]

Step 1. If a geodesic is a plane curve, then it is a line of curvature.

Step 2. If all geodesics of a connected surface S are plane curves, then every point of S is an umbilical point (principal curvatures are same, $k_1 = k_2$).

Step 3. If every point of S is an umbilical point, then S is contained in a plane or a sphere.

Step 1: α geodesic $\Rightarrow$ $N_\alpha = N_S$.

since α plane curve $N' \perp N \Rightarrow N' \parallel T \Rightarrow N'(\alpha(t)) = \lambda(t)\,\alpha'(t)$

$\Rightarrow$ α line of curvature

(α' eigenvector for dN)

Step 2: let $p \in S$. $\forall v \in T_pS$ $\exists$ $\alpha_v \subseteq S$ geodesic with $\alpha_v(0) = p$ and $\alpha_v'(0) = v$.

by Step 1 $\Rightarrow$ $\forall p \in S$ $\forall v \in T_pS$ α_v is a line of curvature $\Rightarrow$ $k_v = k_{v'}$ $\forall v, v'$.

$\Rightarrow$ $k_1 = k_2 = k_p$ $\Rightarrow$ any p is umbilical.

Step 3:

Do Carmo, 3.2 Prop. 4 (Page 147.)