Math 405/538 Differential Geometry
Final Exam

January 20, 2009

1) (10 pts) Let S be a regular surface and $\varphi : U \to S$ be a parametrization with $\varphi(p) = q$. Let the coefficients of the first fundamental form for φ at q are $E = 3$, $F = -2$ and $G = 7$. Let α_1, α_2 be two regular curves in $U \subset \mathbf{R}^2$ with $\alpha_1(0) = \alpha_2(0) = p$ and $\alpha_1'(0) =< 1, 0 >$ and $\alpha_2'(0) =< 1, 1 >$. Compute $\cos\theta$, where θ is the angle between the images of these two curves in S at q.

2) (20 pts) Let $\varphi : \mathbf{R}^2 \to \mathbf{R}^3$ be a parametrization of the torus.

$$\varphi(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u)$$

a. Compute the first and second fundamental form of the torus for φ.

b. Compute the Gaussian curvature $K(u, v)$ for any point $p = \varphi(u, v)$.

c. Decide whether $u = 0$ and $u = \frac{\pi}{2}$ curves are geodesic or not.

3) (20 pts) Let S be a surface in $\mathbf{R}^3$, and α is a regular curve which is a line of curvature in S. Then show that,

a. $N'(t) = \lambda(t)\alpha'(t)$ for any t where $N(t) = N(\alpha(t))$ with N is the unit normal of S.

b. If α is also a geodesic in S, then α is a plane curve. (Hint: Use part a)

4) (15 pts) Let S be the unit sphere centered at origin in $\mathbf{R}^3$, and P be the plane $\{z = \frac{1}{2}\}$. Let α be the curve which is the intersection of S and P, i.e. $\alpha = S \cap P$. Then,

a. Compute the curvature, geodesic curvature and normal curvature of α at any point $p \in \alpha$.

b. Compute the holonomy of α.

5) (15 pts) Prove or give a counterexample: Is S is a surface with Gaussian curvature $K > 0$, then the curvature of any regular curve $\alpha \subset S$ is everywhere positive.

6) a. (10 pts) Write the statements of Global and Local Gauss-Bonnet Theorems. Write the definition of holonomy.

b. (10 pts) Let S be a regular surface in $\mathbf{R}^3$ which is homeomorphic to a disk. Show that for any two points $p, q \in S$, the parallel transport of a vector in T_pS to a vector in T_qS is independent of the path from p to q if and only if the Gaussian curvature is 0 everywhere in S.

(Hint: Use the relation between holonomy and Gaussian curvature)