Lecture #2:

This section is about comparing performance of different algorithms for the same problem. For the sake of concreteness, let us focus on sorting by using comparisons. We have several algorithms: quicksort, insertion sort, heapsort, merge sort, etc. When we take up any algorithm, we measure its performance by the relationship between time taken and the size of the problem. In this instance size is measured by the number of elements in the array that needs to be sorted. However, it may be that for the same algorithm different problems of the same size may take different amounts of time. Moreover, the time may also depend on the processor. To avoid the latter complication, we look at the number of operations. If the algorithm uses several types of operations, we need to have an idea of relative difficulty of each. To take into account the former, there are two common approaches: (i)look at the worst case and (ii) look at the average. In either case we get a functional relation between the size and the measured "complexity" (a surrogate for time taken). It is these functions that are compared.

Since our concern is about large problems, we only look the behaviour for large values of size. Suppose we are interested in comparing two algorithms whose "growth" functions are $f(n)$ and $g(n)$ where n represents the "size" of the instance. Normally, these functions are positive, increasing functions and tend to infinity as $n \to \infty$. We study limiting behaviour of the ratio $h(n) = \frac{f(n)}{g(n)}$ as $n \to \infty$.

Asymptotic Order Notation We study the notations $\Theta, O, \Omega, o, \omega$ in this section. All definitions used here are in the book in Chapter 2. We will give examples of some of them.

Definition 1 $[f(n) = O(g(n))] \iff [h(n) \leq c_2 \forall n \geq n_0$ *for some* $c_2 < \infty$ *and* $n_0]$. *Sometimes it is easier to use the relation* $[\lim_{n\to\infty}[h(n)] = c < \infty] \Rightarrow [f(n) = O(g(n))]$. *But this does not always work. Note the one sided implication.*

Definition 2 $[f(n) = \Omega(g(n))] \iff [h(n) \geq c_1 \forall n \geq n_0$ *for some* $c_1 > 0$ *and* $n_0]$. *Sometimes it is easier to use the relation* $[\lim_{n\to\infty}[h(n)] = c > 0] \Rightarrow [f(n) = \Omega(g(n))]$. *But this does not always work. Note the one sided implication.*

Definition 3 $[f(n) = \Theta(g(n))] \iff [c_1 \leq h(n) \leq c_2 \forall n \geq n_0$ *for some* $c_1 > 0$, $c_2 < \infty$ *and* $n_0]$ *Sometimes it is easier to use the relation* $[\lim_{n\to\infty}[h(n)] = c] \Rightarrow [f(n) = \Theta(g(n))]$ *if* $0 < c < \infty$. *But this does not always work. Note the one sided implication.*

Definition 4 $[f(n) = o(g(n))] \iff [\lim_{n\to\infty} h(n) = 0]$.

Definition 5 $[f(n) = \omega(g(n))] \iff [\lim_{n\to\infty} h(n) = \infty]$

Examples

- $f(n) = an^2 + bn + c; a > 0; g(n) = n^2$. Then $f(n) = \Theta(g(n))$. Only the leading power counts.
- $10^6 n = o(n^2)$. Smaller powers grow slower than larger powers.
- If $f(n) = o(g(n))$ then it follows that $f(n) = O(g(n))$ automatically.
- $10^{-6} n^2 = \omega(n)$
- If $f(n) = O(g(n)) \iff g(n) = \Omega(f(n)$
- If $f(n) = o(g(n)) \iff g(n) = \omega(f(n)$
- If $f(n) = n$ and $g(n) = n^{1+\sin n}$ then we can not say much in comparing the functions asymptotically.
- $n! = o(n^n)$
- If $f(n) = O(n^b)$ for fixed (independent of n) value b, we say that $f(n)$ is polynomially bounded. By definition this means that

  $$f(n) \leq cn^b \text{ for some } c < \infty \text{ for } n \geq n_0 \text{ for some } n_0.$$

  Hence it follows that

  $$\lg f(n) = b \lg n + \lg c \leq 2b \lg n \text{ for } n \geq n_0 \text{ for some } n_0$$

  Hence $[f(n) = O(n^b)] \Rightarrow [\lg f(n) = O(\lg n)]$. I leave proving the converse to you.
- If $f(n) = n^b$ and $g(n) = a^n$ with $a > 1$, then $f(n) = o(g(n))$. This says that any polynomially growing function is smaller than an exponentially growing function asymptotically. This is an important result and so we will show its proof.

  Let $h(n) = \frac{f(n)}{g(n)} = \frac{n^b}{a^n}$. We show that $\lim_{n\to\infty} h(n) = 0$ by showing that $\frac{h(n+1)}{h(n)} < 1$ for $n \geq n_1$.

  $\frac{h(n+1)}{h(n)} = (\frac{n+1}{n})^b \frac{1}{a}$ and hence the result follows.
- If $f(n) = \lg^b n = (\lg n)^b$ and $g(n) = n^a$ with $a > 0$, then $f(n) = o(g(n))$. This says that any power of logarithmically growing function is smaller than a polynomially growing function asymptotically. In particular, $\lg n = o(n^\epsilon)$ for all $\epsilon > 0$.

  By the previous result, we know that $\lim_{m\to\infty}[\frac{m^b}{c^m}] = 0$ when $c > 1$. Let $c = 2^a, a > 0$ and $m = \lg n$. $c^m = (2^a)^{\lg n} = 2^{a \lg n} = (2^{\lg n})^a = n^a$ Hence this result follows.

- $\lg[n!] = \sum_{i=1}^{n} \lg i = \Theta(n \lg n)$.

  $\sum_{i=1}^{n} \lg i \leq \sum_{i=1}^{n} \lg n = n \lg n$. Hence we may choose $c_2 = n_0 = 1$ in showing $\sum_{i=1}^{n} \lg i = O(n \lg n)$.

  $\sum_{i=1}^{n} \lg i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lg i \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lg \frac{n}{2} = \lfloor \frac{n}{2} \rfloor \lg \frac{n}{2} \geq \frac{n}{2} \lg \frac{n}{2} - \lg \frac{n}{2} = \frac{n}{2} \lg n - \frac{n}{2} - \lg n + 1 = (n \lg n)[\frac{1}{2} - \frac{\frac{n}{2} - \lg n + 1}{n \lg n}]$

  $\lim_{n \to \infty} [\frac{1}{2} - \frac{\frac{n}{2} - \lg n + 1}{n \lg n}] = \frac{1}{2}$ Hence $[\frac{1}{2} - \frac{\frac{n}{2} - \lg n + 1}{n \lg n}] > \frac{1}{4}$ for sufficiently large n. Hence $\sum_{i=1}^{n} \lg i = \Omega(n \lg n)$.

  These two together give what we want.

The last three results are very important.

Some Relations:

- $[f(n) = O(g(n))] \Rightarrow [g(n) = \Omega(f(n))]$
- $[f(n) = \Theta(g(n))] \Leftrightarrow [g(n) = \Theta(f(n))]$
- $[f(n) = \Theta(g(n)) \Longleftrightarrow [\{f(n) = O(f(n))\}$ **and** $\{f(n) = \Omega(g(n))\}]$
- $[f(n) = o(g(n))] \Rightarrow [f(n) = O(g(n))]$
- $[f(n) = \omega(g(n))] \Rightarrow [g(n) = o(f(n))] \Rightarrow [g(n) = O(f(n))] \Rightarrow [f(n) = \Omega(g(n))]$
- We say a subset of $\{\Theta, O, \Omega, o, \omega\}$ is a compatible set if there are functions $f(n)$ and $g(n)$ that satisfy all relations in the subset. From above bullet items, we see that $\{o, O\}, \{w, \Omega\}, \{O, \Omega, \Theta\}$ are compatible. The sets $\{o, \Omega\}, (o, \Theta\}, \{o, \omega\}, \{\omega, O\}, \{\omega, \Theta\}$ are not compatible. This is sometimes used to show that $f(n) \neq O(g(n))$ etc.