Some more NP-C results:

**HW#7:**

**PARTITION**

**Input:** Positive integers $x_1, x_2, \ldots, x_n$

**Q:** Is there a subset $S \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in S} x_i = \sum_{i \notin S} x_i?$$

[Recall numbers are given in binary].

1. **PARTITION \notin NP**

Verification Certificate for "yes" instance:

$S$ itself given as a 0/1 vector of size $n$.

**Verification Alg.** [Poly time alg]

1. Add $\sum_{i \in S} x_i$ in binary (poly in # of bits)

2. Check $\sum_{i \in S} x_i = \sum_{i \notin S} x_i$ in binary (poly in # of bits)

Attempt to give "false" certificates will be "found out" by the above algorithm.
2. \textsc{Partition ENP-C}

\[ ?^{\text{ENP}}_P \leq_p \text{PARTITION} \]

\[
\downarrow \quad \text{SUBSET-SUM} \quad \downarrow
\]

\[ \text{Arbitrary instance} \quad x_1, \ldots, x_n \geq t \]

\[ ? 
\]

Q: Is there \( S \subseteq \{1, 2, \ldots, n\} \)

\[ \sum_{i \in S} x_i = t \quad ? \]

Corresponding instance (to be constructed)

\[ x_1, x_2, \ldots, x_n, t+1, \]

\[ \sum_{i=1}^{n} x_i - t + 1 \]

(Poly size in Subset problem size.)

\[ \text{Need to show} \quad \{\exists S \text{ such that } \sum_{i \in S} x_i = t \} \quad \Rightarrow \text{yes} \]

\[ \Rightarrow \quad \text{Suppose \textsc{SUBSET-SUM is a "yes" instance}} \]

\[ \exists S \text{ such that } \sum_{i \in S} x_i = t \]

\[ \text{For \textsc{Partition}}: \]

\[ \{ S, \frac{1}{2} \sum_{i=1}^{n} x_i - t + 1 \} \]

\[ \{ \frac{1}{2} \sum_{i=1}^{n} x_i - t + 1, \}

\[ \sum_{i \in S} x_i + \sum_{i \in S} x_i - t + 1 \]

\[ \frac{1}{2} \sum_{i \in S} x_i + t + 1 \]

\[ \frac{1}{2} \sum_{i \in S} x_i + 1 \]
In PARTITION instance "created", if it yes instance, then \( t + 1 \), and \( \sum_{i=1}^{n} x_i - t + 1 \) CANNOT be on the same side since sum of these two \( \sum_{i=1}^{n} x_i + 2 \) and all the rest add up to only \( \sum_{i=1}^{n} x_i \). Hence these must be on opposite sides.

Let those with \( \sum_{i=1}^{n} x_i - t + 1 \) be called \( S \leq \{1, 2, \ldots, n\} \).

Hence two sides are

\[
\begin{align*}
S & \quad \overline{S} \\
\sum_{i=1}^{n} x_i - t + 1 & \quad t + 1 \\
\sum_{i \in S} x_i + \sum_{i \in \overline{S}} x_i - t + 1 & = \sum_{i \in S} x_i + t + 1 \\
\sum_{i \in S} x_i + \sum_{i \in \overline{S}} x_i - t & = \sum_{i \in S} x_i + t \\
2 \sum_{i \in S} x_i & = 2t \\
\therefore \text{subset sum is yes}
\end{align*}
\]
0/1 INTEGER-PROGRAM

Input: An integer matrix \( \mathbf{A} \), integer vector \( \mathbf{b} \).

Q: Is there a 0/1 vector \( \mathbf{x} \) such that

\[
\mathbf{A} \mathbf{x} \geq \mathbf{b} \quad \left( \sum_{j=1}^{m} a_{i,j} x_j \geq b_i, \ i=1, \ldots, m \right)
\]

1. 0/1 INT. PROG \( \in \text{NP} \)

   V. Certificate: \( \mathbf{x} \) itself (polynomial size)

   "yes" instance

   V. alg.: Check (\( \ast \ast \)) using binary arithmetic (Poly time)

2. 0/1 INT. PROG \( \in \text{NP} \- \text{C} \)

   \( \text{NP} \- \text{C} \subseteq \text{P} \)

   \( \text{3-SAT} \)

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3-SAT

Aub. instance

\[ \phi = C_1 \land C_2 \land \ldots \land C_k \]

\[ C_j = (l_j^{0}, v_j^{1}, v_j^{2}, v_j^{3}) \]

\[ l_j^m \in \{ x, \overline{x}, \ldots, x_n, \overline{x_n} \} \]

Q: Is there \( x \in \{0, 1\}^n \) such that \( \phi = 1 \) ?

For each clause \( C_j = (l_j^{0}, v_j^{1}, v_j^{2}, v_j^{3}) \)

Create an inequality in INT-PROG as follows:

\[ l_j^{0} + l_j^{1} + l_j^{2} \geq 1 \]

Example

\[ C_j = (x_1, v_2, v_3) \]

\[ \downarrow \]

\[ x_1 + (1-x_2) + (1-x_3) \geq 1 \]

Yes, \( \rightarrow \) yes is "clear".

This also works if \( x \) can be arbitrary integer vector in INT-PROG. But showing this \( \in \) in NP is more difficult.
Two More HAM* Reduction

A) \[ \text{UNDIR-HAM-CYCLE} \leq_p \text{DIR-HAM-CYCLE} \]

\[ \text{instance: Undir. } G = [V, E] \]

Q: Is there a \text{HAM-cycle} \( G \)?

\[ \text{instance Dir. graph } H = [U, F] \]

Q: Is there a dir. \text{HAM-CYCLE in } H ?

\[ U = V ; \]

Each edge in \( G \) \[ \rightarrow \]

two edges in \( H \)

Exercise: Show \( \text{yes } \leftrightarrow \text{yes} \) (Exercise for you)

B) \[ \text{DIR-HAM-CYCLE} \leq_p \text{UNDIR-HAM-CYCLE} \]

\[ \text{instance } G = [V, E] \]

\[ \text{dir} \]

\[ ? \]

\[ \text{instance undir. } H = [U, F] \]

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$|U| = 3|V|$

i.e. each vertex in $G$ is replaced by 3 vertices in $H$.

$|F| = |E| + 2|V|$: Construction below.

$\implies$

$\text{HAM-CYCLE in } G$

$\implies$

$\text{HAM-CYCLE in } H$
HAMCycle in H

Since "second" copy of each node is "only" connected to its own first & third copies, these nodes must occur together in any HAM-cycle in H.

Moreover, if for some node they occur in the order \( A' \rightarrow A_2 \rightarrow A_3 \)

Then this same order must occur for all nodes. Hence we can use some two "picture" to show

\[ \text{yes} \iff \text{yes} \]