All pairs Shortest Paths: Ch 25 of CLRS

Here origin is not fixed. Notation first:

\[ \delta(i,j) = \text{the length of a SP from } i \text{ to } j \]

(directed version)

\[ \delta(i,j) \quad \text{i} \quad \text{to} \quad \text{j} \]

Matrix \( D \) of size \( V \times V \)

\[ D = V \times V \]

\( (i,j) \) element in \( D \) is \( \delta(i,j) \) or its current estimate

\( (i,j) \quad \quad \Pi \leftrightarrow \text{see above picture} \)

Why can we not repeat Single Origin alg with change of origins?

Complexity

DAG repeated \(|V|\) times:

\[ \Theta((E+V)V) \]

\[ = \Theta(EV) \]

\[ = \Theta(V^3) \]

in dense graphs since \( |E| = \Theta(V^2) \)

Dijkstra repeated \(|V|\) times:

\[ \Theta(1E1V \cdot \log |V|) \]

\[ = \Theta(V^2 \log |V|) \]

for dense graphs.
Bellman Ford repeated \( |V| \) times:
\[
\Theta (|E||V|^{2}) = \Theta (|V|^4)
\]
for dense graphs is not acceptable!

Want \( \Theta (|V|^4) \)
so first we give an alg. with \( \Theta (|V|^3 \log |V|) \)
And then modify to one with \( \Theta (|V|^3) \)

There are two methods. First of these is due to Warshall called **Matrix - Min - Addition**. We explain this operation first in general and then show how it applies to SP.

\[
\begin{align*}
\text{Regular Matrix - Multiplication:} & \quad \mathbf{A} \cdot \mathbf{B} = \mathbf{C} \\
C_{ij} &= \sum_{k=1}^{n} A_{ik} \cdot B_{kj} \\
&= \Theta (mnp) \\
\text{if } m=n=p, \Theta (n^3) \\
\end{align*}
\]

\[
\begin{align*}
\text{Matrix - Min - Addition} & \quad \mathbf{A} +_{\text{m}} \mathbf{B} = \mathbf{C} \\
C_{ij} &= \min_{k=1}^{n} \{ q_{ik} + b_{kj} \} \\
&= \Theta (mnp) \text{ (though different operations)} \\
\text{if } m=n=p, \Theta (n^3) \\
\end{align*}
\]

Any code for multiplication easily modified for this.
Wasshull Algorithm:

Input: A $|V| \times |V|$ matrix $W$ whose entries $W_{i,j}$: edge weight of edge $(i \rightarrow j)$

$W_{i,i} = 0$.

At no edge $(i \rightarrow j)$ in the graph, $W_{i,j} = \infty$.

Example:

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & \infty & -4 \\
2 & \infty & \infty & \infty & \infty & 17 \\
3 & \infty & \infty & \infty & \infty & \infty \\
4 & \infty & \infty & \infty & \infty & \infty \\
5 & \infty & \infty & \infty & \infty & \infty \\
\end{array} \]

$(G, w)$

The matrix and the graph contain the same information and one can be obtained from the other. We will generate matrices $L^{(1)}$, $L^{(2)}$, ..., $L^{(\infty)}$ where $|V| = n$ with elements of $L^{(k)}$: $L_{i,j}^{(k)}$ = the length of a SP from $i$ to $j$ using $k$ or fewer edges.

This is done using the formula: $L^{(0)} = W$

$L^{(k)} = L^{(i)} + L^{(k-i)}$ for $k \geq 2$

Important: It does not matter which value $i$ is used.
Sample Calculation

\[ l_{4,2}^{(2)} = \min \left\{ l_{4,1}^{(1)} + l_{1,2}^{(1)}, \frac{l_{4,2}^{(1)} + l_{22}^{(1)}}{4,3}, \frac{l_{4,3}^{(1)} + l_{3,2}^{(1)}}{4,5} \right\} \]

\[ = \min \left\{ 2+3, \infty + 0, -5+4, 0 + \infty, \infty + \infty \right\} \]

\[ = -1 \]

**[\Pi_{i,j}] Calculation:**

\[ \Pi_{i,i}^{(1)} = \begin{array}{ccc}
-1 & 1 & -1 \\
-1 & -2 & 2 \\
3 & -1 & -1 \\
4 & -4 & -1 \\
-1 & -1 & -5 \\
\end{array} \]

\[ [\Pi_{i,j}]^{(2)} = [\Pi_{i,j}]^{(1)} [3,2] \leftarrow 3 \]

This path: \( 4 \rightarrow 3 \rightarrow 2 \) = -1

\[ \Theta(V^3 \cdot \log V) \]

If \( c_{ii}^k < 0 \) for any \( i \), STOP; Neg cycle

Else \( L^{(n)} \), \( \Pi^{(n)} \) represent SP lengths and the last node on the SP.

Complexity: \( L^{(1)} \rightarrow L^{(2)} \rightarrow L^{(3)} \rightarrow L^{(8)} \ldots \)
Floyd-Warshall Algorithm

We create a sequence \((D^0, T^0)\), \((D^1, T^1)\) \(\ldots\) \((D^n, T^n)\)

As follows:

\[ D^0 = W; \quad T^0 = \]

\[
\begin{array}{ccc}
- & 1 & 1 \\
- & - & 2 & 2 \\
- & 3 & - & - \\
4 & - & 4 & - \\
- & - & - & 15 \\
\end{array}
\]

Suppose we know \((D^{k-1}, T^{k-1})\); To get \((D^k, T^k)\):

\[ d_{ij}^{(k)} = \min \{ d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \} \]

\[ T_{ij}^{(k)} = \begin{cases} T_{ij}^{(k-1)} & \text{if } d_{ij}^{(k)} = d_{ij}^{(k-1)} \\ T_{ik}^{(k-1)} & \text{if } d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases} \]

Complexity: \(\Theta(n^3)\)

Prog of correct ren for both algorithms on the next page
Matrix Min-addition

Claim: $l_{ij}^{(k)}$ represents the length of a SP from $i$ to $j$ using $k$ or fewer edges (assuming no neg. cycles so far)

Pf:

Let a SP from $i$ to $j$ using $k$ or fewer edges look like

![Graph Diagram]

Break this into two parts: one with no more than $r$ edges, and the other (hence) with no more than $(k-r)$ edges.

Without loss, each piece can be assumed to be a SP with no more than appropriate number of edges.

Hence first piece: $l_{i,p}^{(r)}$

Second: $l_{p,j}^{(k-r)}$

By I.H.

$$l_{i,j}^{(k)} = \min \left\{ l_{i,p}^{(r)} + l_{p,j}^{(k-r)} \right\}$$

Hence the proof. If $l_{i,i}^{(k)} < 0$: neg. cycle.
Floyd-Warshall Proof

\[ d_{i,j}^{(k)} = \text{the length of a SP from } i \text{ to } j \text{ among paths which use only nodes } \{1, 2, \ldots, k\} \text{ as intermediate nodes in any order.} \]

Nodes are numbered \(\{1, 2, \ldots, n\}\) Arbitrarily.
No node has number 0.
Node numbered 1. (Not just one node.)

These are the paths considered.

And so on. So \(d_{i,j}^{(k)}\) looks like

\[ d_{i,j}^{(k)} = d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)} \leq k-1 \]

or

\[ d_{i,j}^{(k)} \leq k-1 \]

\[ d_{i,j}^{(k-1)} \leq k-1 \]

\[ d_{i,k}^{(k-1)} \leq k-1 \]

\[ d_{k,j}^{(k-1)} \leq k-1 \]

R occurs only once.

(No neg. cycle)