Solution to Assignment #2:

1. (a) (4.3-6) Show that \( t(n) = O(n \lg n) \) for the relation:

\[
t(n) = 2t\left(\left\lfloor \frac{n}{2} \right\rfloor + 17\right) + n
\]

Solution: Let \( m = n - 34 \). This implies that \( \left\lfloor \frac{n}{2} \right\rfloor + 17 = \left\lfloor \frac{m}{2} \right\rfloor + 34 \).

Hence our equation becomes:

\[
t(m + 34) = 2t(m) + (m + 34)
\]

\[
s(m) = 2s\left(\left\lfloor \frac{m}{2} \right\rfloor \right) + \Theta(m)
\]

\[
= \Theta(m \lg m)
\]

if we let \( S(m) = t(m + 34) \). Hence,

\[
t(n) = t(m + 34) = s(m) = \Theta(m \lg m) = \Theta((n - 34) \lg(n - 34))
\]

Now we show that \((n + c) \lg(n + c) = \Theta(n \lg n)\) and this will complete the proof.

\[
\lim_{n \to \infty} \frac{(n + c) \lg(n + c) - n \lg n}{n \lg n}
\]

\[
= \lim_{n \to \infty} \frac{(\lg(n + c) - \lg n)}{\lg n} + \lim_{n \to \infty} \frac{c \lg(n + c)}{n \lg n}
\]

\[
= \lim_{n \to \infty} \frac{\lg(1 + \frac{c}{n})}{\lg n} = 0
\]

Hence

\[
\lim_{n \to \infty} \frac{(n + c) \lg(n + c)}{n \lg n} = 1
\]

and therefore, \((n + c) \lg(n + c) = \Theta(n \lg n)\).

(b) 4.3-9: Solve the relation \( t(n) = 3t(\sqrt{n}) + \log n \)

Solution:

Let \( \lg n = m; n = 2^m \). The above equation changes to

\[
t(2^m) = 3t(2^{\frac{m}{2}}) + m
\]

Now let \( t(2^x) = s(x) \) and the equation changes to

\[
s(m) = 3s(\frac{m}{2}) + m
\]

This equation is solved by master theorem (case 1) to yield the solution \( s(m) = \Theta(m^{\log_2 3}) \). Hence the solution of the original equation is \( t(n) = \Theta((\lg n)^{\log_2 3}) \).
2. **4.4-2**: Use a recursion tree to give an asymptotically tight solution to the relation:

\[ t(n) = t\left(\frac{n}{2}\right) + n^2 \]

Use substitution method to verify your answer.

\[
\begin{align*}
& n^2 \\
& \quad \downarrow \\
& (n/2)^2 \\
& \quad \downarrow \\
& (n/4)^2 \\
& \quad \downarrow \\
& (n/2^k)^2
\end{align*}
\]

where \( k = \log n \). Adding we get

\[
t(n) = n^2 \sum_{j=0}^{k} \left(\frac{1}{2}\right)^j = \Theta(n^2).
\]

**Verification**: By using induction hypothesis on smaller values we get

\[
t(n) \leq n^2 + c(n/2)^2 = n^2[1 + \frac{c}{4}] \leq cn^2 \text{ if } c > \frac{4}{3}
\]

**4.4-6**: Use a recursion tree to give an asymptotically tight solution to the relation:

\[ t(n) = t(\alpha n) + t((1 - \alpha)n) + cn \]
where \( c > 0 \), and \( 0 < \alpha < 1 \) are given constants.

The depth of the tree is \( \frac{\lg n}{\lg \alpha} = \Theta(\lg n) \). Hence,

\[ t(n) = \Theta(n \lg n) \]

Hence \( t(n) = \Omega(n \lg n) \).

3. Problem 4-3: (b),(c),(f)

b) \( t(n) = 3t(\frac{n}{3}) + \frac{n}{\lg n} \)

\( f(n) = \frac{n}{\lg n} \); \( n^{\log_3 a} = n \). So both cases 2 and 3 do not apply. Moreover, \( \lim_{n \to \infty} [\frac{n^{1-\varepsilon}}{f(n)}] = 0 \) and hence \( f(n) = \omega(n^{1-\varepsilon}) \). So we have \( f(n) \neq O(n^{\log_3 a - \varepsilon}) \). So the master theorem does not apply in this case. This equation is of the form \( t(n) = at(\frac{n}{3}) + n^{\log_3 a}(\lg n)^k; \) \( b > 1; a \geq 1 \) but \( k \geq 0 \). In this case \( k = -1 \). So we try iteration method after changing variables.

Let \( \log_3 n = k; n = 3^k; t(3^k) = s(k) \) and the equation becomes:

\[ t(n) = s(k) = 3s(k - 1) + \frac{3^k}{ck} \]

\[ = \frac{3^k}{ck} + 3 \cdot \frac{3^{k-1}}{c(k - 1)} + \ldots \]

\[ = \frac{1}{c} \sum_{i=1}^{k} \frac{1}{i} \]

\[ = \Theta(3^k \lg k) \]

\[ = \Theta(n \lg \lg n) \]

Here \( \lg n = \frac{\log_3 n}{\log_3 3}; c = \frac{1}{\log_3 3}. \)
4. (4-4) (page 108) Fibonacci Sequence:

\[ F_0 = 0 \]
\[ F_1 = 1 \]
\[ F_i = F_{i-1} + F_{i-2} \quad i \geq 2 \]

(a) \[
\mathcal{F}(z) = \sum_{i=0}^{\infty} F_i z^i = z + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) z^i
\]
\[ = z + z \sum_{i=0}^{\infty} F_i z^i + z^2 \sum_{i=0}^{\infty} F_i z^i
\]
\[ = z + z \mathcal{F}(z) + z^2 \mathcal{F}(z)
\]
\[ = \frac{z}{1 - z - z^2}
\]
\[ = \frac{z}{(1 - \phi z)(1 - \hat{\phi} z)}
\]

the last of these follows from the fact that \( \frac{1}{\phi} \) and \( \frac{1}{\hat{\phi}} \) are the roots of \( [1 - z - z^2 = 0] \).

(b) 
\[
\frac{z}{(1 - \phi z)(1 - \hat{\phi} z)} = \frac{1}{\sqrt{5} \phi} \left( \frac{1}{(1 - \phi z)} - \frac{1}{(1 - \hat{\phi} z)} \right)
\]
\[
\frac{1}{(1 - \phi z)} = \sum_{i=0}^{\infty} \phi^i z^i \quad \text{and} \quad \frac{1}{(1 - \hat{\phi} z)} = \sum_{i=0}^{\infty} \hat{\phi}^i z^i.
\]

Hence

(c) 
\[
\mathcal{F}(z) = \frac{1}{\sqrt{5} \phi} \sum_{i=0}^{\infty} (\phi^i - \hat{\phi}^i) z^i
\]

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(d) Hence
\[ F_i = \frac{1}{\sqrt{5}}(\phi^i - \phi^i) \]
and \( |\phi^i| < 1 \). Hence \( F_i = \left[ \frac{\phi^i}{\sqrt{5}} \right] \).

(e) Want to show that \( F_{i+2} \geq \phi^i \) for \( i \geq 0 \)

We do this by induction. Clearly true for \( i = 0 \). Suppose it is true for \( i \leq k \) Will show for \( i = k + 1 \).

\[ F_{k+3} = F_{k+1} + F_{k+2} \geq \phi^{k-1} + \phi^k = \phi^{k+1} \left[ \frac{1}{\phi^2} + \frac{1}{\phi} \right] = \phi^{k+1} \left[ \frac{\phi + 1}{\phi^2} \right] = \phi^{k+1}. \]

\[
\frac{1}{\phi^2} = 1 - \frac{1}{\phi} \\
\phi^2 = 1 + \phi \\
\left[ \frac{\phi + 1}{\phi^2} \right] = 1
\]

(Recall \( \frac{1}{\phi} \) is a solution of the equation
\[ 1 - z - z^2 = 0 \]

The result now follows.