CS6381: Lecture #11 9.15.2020

Minimum Cost Flows:

One version is due to Busacker & Gowan, and Jewell's algorithm. Consider:

$$0 \le f_{i,j} \le u_{i,j} \quad \forall (i,j) \in E$$

$$\sum_j f_{ij} - \sum_j f_{j,i} = \begin{cases} F & i = s \\ 0 & i \neq s, t \\ -F & i = t \end{cases}$$

$$\text{Min} \sum_i \sum_j c_{ij} f_{ij}$$

<u>Input data</u>: $\{u_{ij}, c_{ij}\}_{(i,j) \in E}$; F.,

Given a feasible flow f^0, define <u>residual graph</u> as follows:

$u_{ij}^{f^0}$ = residual capacity of (i,j) in the direction $i \to j$

$$= u_{ij} - f_{i,j}^0 \cancel{+ f_{j,i}^0} \quad (i,j) \in E$$

$\cancel{(\text{if } f_{j,i}^0 > 0},$

reverse $u_{ji}^{f^0}$: $f_{i,j}^0$ if $(i,j) \in E$, $f_{ij}^0 > 0$

~~If $f^0_{ij} > 0$, and $(i,j) \in E(G)$~~,

for $(i,j) \in E$: E^{f^0} has possibly two edges (i,j) & (j,i)

$$u^{f^0}_{ij} = u_{ij} - f^0_{ij} \quad : \quad c^{f^0}_{ij} : c_{ij}$$

$$u^{f^0}_{ji} = f^0_{i,j} \quad : \quad c^{f^0}_{ji} = -c_{ij}$$

Edges with 0 residual capacity are removed.

An s.t path is <u>flow augmenting</u> if all edges directed in the forward direction (as you traverse from s to t) have +ve residual capacity and all reverse edges have positive flow in the forward direction ; i.e this is a <u>directed</u> s-t path in G^{f^0}. The total cost of such a path is the sum of costs in the residual graph (recall for edges arising as "reverse" edges costs are $-c_{ij}$)

Cost of a cycle in G^f is similarly Calculat

Lemma 1: A feasible flow f^0 is optimal if and only if it admits no negative cycle in G^{f^0}.

Proof: The only if part is obvious. For the if part, note that difference between two feasible solutions is a sum of cycles flows. If one has less cost, Then one of these cycles has negative cost ~~is~~ in G^{f^0} and hence the result.

If $\{f_{ij}\}$ and $\{g_{ij}\}$ are feas. solution then

$$\sum_j \underbrace{(f_{ij} - g_{ij})}_{h_{ij}} - \sum_j \underbrace{(f_{ji} - g_{ji})}_{h_{ji}} = 0 \;\forall i \in$$

$$\sum_j h_{ij} - \sum_j h_{ji} = 0 \quad \forall i.$$

h: may be +ve, −ve or 0.

If $h_{ij} > 0$ imagine in a different graph, edge going from i to j with edge wt $= h_{ij}$

If $h_{ij} < 0$ imagine an edge going from j to i with edge wt $= -h_{ij}$. In this wtd graph, weight is conserved: at each node sum of outgoing weights = sum of incoming weights.

Hence, this graph contains dir. cycles with positive edge weights. We can decompose (like path decomposition) this weighted graph into cycles with positive weights

If $\sum_j \sum_i c_{ij} f_{ij} < \sum_i \sum_j c_{ij} g_{ij}$,

then $\sum_j \sum_i c_{ij} h_{ij} < 0$

∴ One of the above cycles has negative weight and such a cycle can be found using FW alg. on the residual graph.

Hence, if f^0 is optimal, residual graph can not have negative cycles. Completing the proof of the lemma

<u>Theorem</u>: Let f^0 be an optimal flow for F^0

Now consider the residual graph G^{f^0}

(with costs: c_{ij} for $(i,j) : f^0_{ij} < u_{ij}$,

Capacities $u_{ij} - f^0_{ij}$

& $-c_{ij}$ for $(j,i) : f^0_{ij} > 0$

Capacity: f^0_{ij}.

Find a s-t path $P_{s,t}$ with min cost (using SP. alg.) (Recall: residual graph has no negative cycle) since f^0 is opt for F^0.

Suppose residual capacity of $P_{s,t}$ is δ (all paths have positive residual capacity)

Then, an optimal solution for $F = F^0 + \epsilon$ is given by augmenting f^0 by ϵ on $P_{s,t}$ for all $0 \le \epsilon \le \delta$.

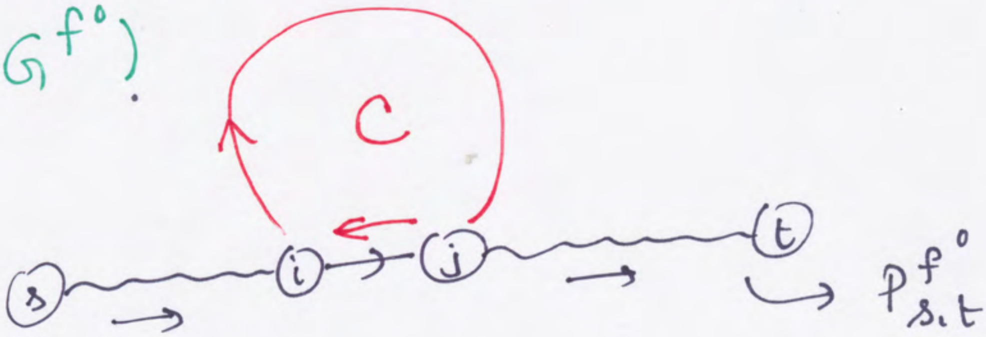
This is Busacker, Gowen Jewell etc alg

Remark: Using this result, we can build optimal solution for all F: $0 \le F \le F_{max}$.

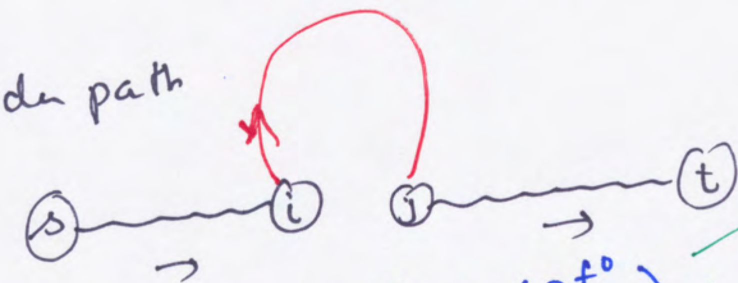
Proof of Theorem:

Let f^0 be the flow before augmentation and
f^1 " " after "

By the lemma, if f^1 is not optimal for $F^0+\epsilon$ then there is a negative cycle C in G^{f^1}. Since f^0 is optimal for F^0, ∄ negative cycle in G^{f^0}.
Hence C must contain an edge (j,i) such that (i,j) was on $P^{f^0}_{s,t}$. (min-cost path in SP G^{f^0}).



Consider path



Note $\text{cost}(i,j)$ cancels out

Its total cost $= \text{Cost}(P^{f^0}_{s,t}) + \text{Cost}(C) \cancel{+\text{cost}(i,j)} \cancel{-\text{cost}(i,j)}$

$< \text{Cost}(P^{f^0}_{s,t})$ since $\text{Cost}(C) < 0$

This contradicts $P^{f^0}_{(s,t)}$ is SP (min-cost path) in G^{f^0}. Hence the result.

Algorithm:

Step.0: Starting with $f \equiv 0$, Saturate all negative cycles in the respective residual graphs (Find these by say F-W- alg) to get an optimal f^0 for $F^0 = 0$.

Step 1: For any optimal pair (f, F) find ~~the~~ a min-cost path in G^f from s to t and saturate it to get an optimal pair for $(f', F' = F + \Delta)$ where Δ is the maximum augmentation on this path. (If you need $< F+\Delta$, stop at that point).

Go back to Step 1 with (f', F').
unless $f' = F_{max}$.

Complexity: $O(n^3 F) \rightarrow$ not polynomial

but can be made polynomial by scaling techniques.

The shortest paths can be done using Dijkstra algorithm – see Edmonds-Karp paper. (Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems, J.ACM, 19 (1972) pp 248-264.) – A MUST READ

Algorithm Based on LP duality

Consider:

$$l_{i,j} \le f_{i,j} \le u_{i,j} \qquad \forall (i,j) \in E$$

(P)
$$\sum_j f_{i,j} - \sum_j f_{j,i} = \left\{ q_i \quad i \in V. \right.$$

$$\text{Min} \sum_{(i,j)\in E} c_{ij} f_{ij}$$

$$x_{i,j} \geq 0 : \quad f_{ij} \geqq l_{ij}$$ – Dual variables

$$y_{ij} \geqq 0 : \quad -f_{ij} \geqq -u_{ij}$$

$$\pi_i : \quad \sum_j f_{ij} - \sum_j f_{j,i} = q_i$$

unrestd

$$\text{Min} \sum_{(i,j)\in E} c_{ij} f_{ij}$$

D
$$\begin{cases} x_{i,j} \geqq 0, \quad y_{i,j} \geqq 0, \quad \pi_i : \text{unrestd} \\ \pi_i - \pi_j + x_{i,j} - y_{i,j} = c_{i,j} \quad \forall (i,j) \in E \\ \text{Max} \sum_{i \in V} q_i \pi_i + \sum_{(i,j) \in E} \left[l_{ij} x_{ij} - u_{ij} y_{i,j} \right] \end{cases}$$

Complementary Slackness Conditions

$$f_{ij}^* > l_{ij} \Rightarrow x_{ij}^* = 0 \; ; \; x_{ij}^* > 0 \Rightarrow f_{i,j}^* = l_{ij}$$

$$f_{i,j}^* < u_{ij} \Rightarrow y_{i,j}^* = 0 \; ; \; y_{i,j}^* > 0 \Rightarrow f_{ij}^* = u_{ij}$$

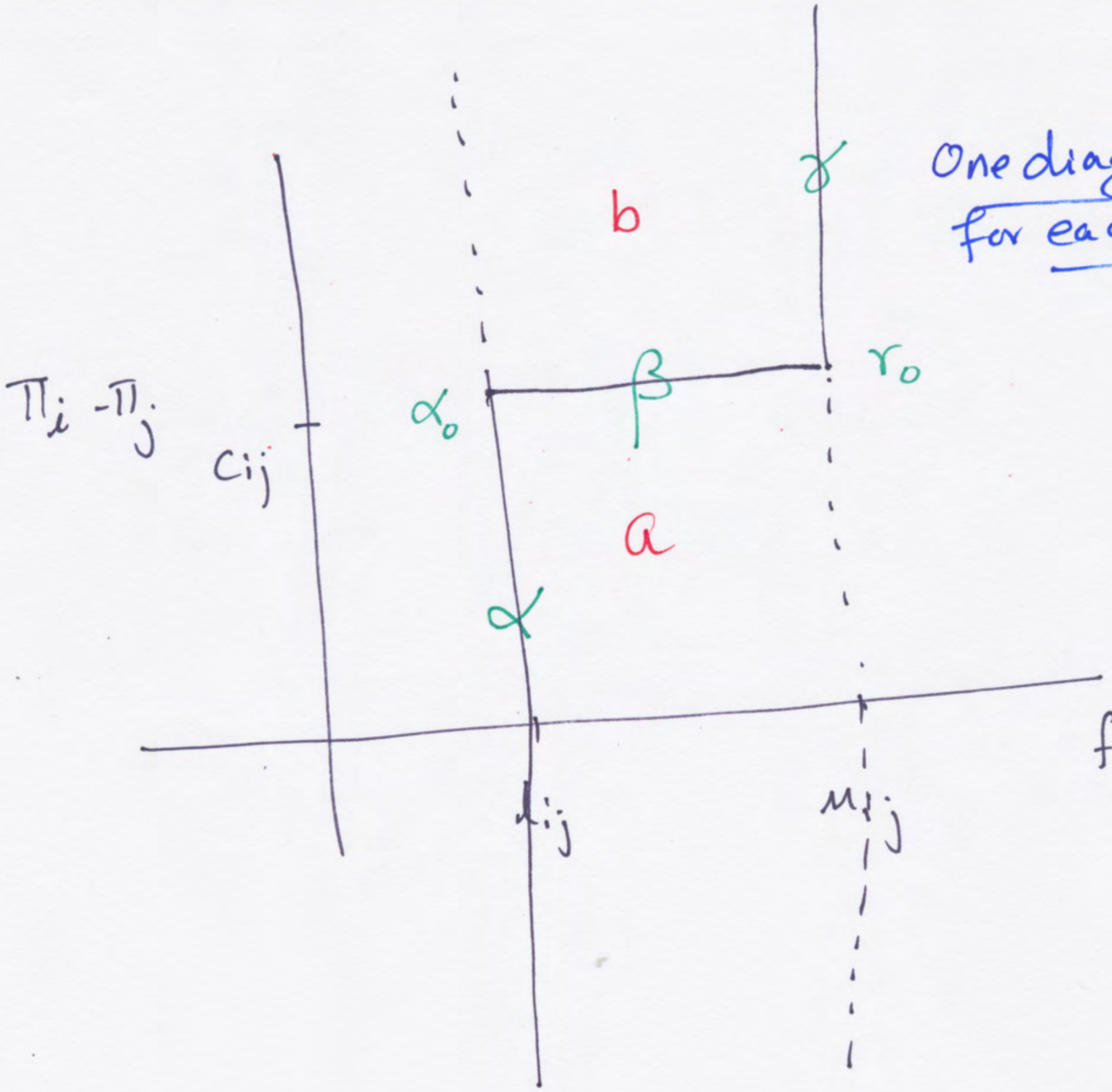
Hence; $\pi_i^* - \pi_j^* < c_{ij} \Rightarrow x_{i,j}^* > 0 \Rightarrow f_{ij}^* = l_{ij}$

$\pi_i^* - \pi_j^* > c_{ij} \Rightarrow y_{i,j}^* > 0 \Rightarrow f_{i,j}^* = u_{ij}$

$l_{ij} < f_{ij}^* < u_{ij} \Rightarrow x_{ij}^* = y_{i,j}^* = 0 \Rightarrow \pi_i^* - \pi_j^* = c_{ij}$

All of this is captured by what are called Complementary Slackness diagram shown on next page

One diagram for each edge



States	flow	$\pi_i - \pi_j$
β	$\in (l_{ij}, u_{ij})$	c_{ij}
α	l_{ij}	$< c_{ij}$
α_0	l_{ij}	c_{ij}
γ	u_{ij}	$> c_{ij}$
γ_0	u_{ij}	c_{ij}
a	$\in (l_{ij}, u_{ij}]$	$< c_{ij}$
b	$\in [l_{ij}, u_{ij})$	$> c_{ij}$

$\beta, \alpha, \alpha_0, \gamma, \gamma_0$: in-kilter i.e OK for CS condition

a, b: out-of-kilter i.e not OK for CS conditions