Edmonds-Karp Scaling on Out-of-Kilter Method (197x)

This was the first result in producing a (weakly) polynomial algorithm for Min-Cost-Flow. It is in the same paper that includes the now well known E-K algorithm for max-flow which is strongly polynomial.

Here we use another standard form for Min-Cost flow problem. <u>In this formulation all $q_i = 0$.</u> First we show how to convert to this form.

Recall our previous formulation:

$$l_{ij} \le f_{ij} \le u_{ij} \quad \forall (i,j) \in E$$

$$\sum_{j:\,(i,j)\in E} f_{i,j} - \sum_{j:\,(j,i)\in E} f_{j,i} = q_i \quad i \in V \quad (**)$$

$$\text{Min} \sum_{(i,j)\in E} c_{ij} f_{ij}.$$

If we add up all equations in (**), the right side is $\sum_{i\in V} q_i$; left side is $\sum_i \sum_j f_{ij} - \sum_i \sum_j f_{j,i} = 0$

since each f_{ij} appears as $+f_{ij}$ in equ for i & $-f_{ij}$ in " " j

We illustrate this with $n=4$

$i=1$: $\cancel{f_{1,2}} + \cancel{f_{1,3}} + \cancel{f_{1,4}} - \cancel{f_{2,1}} - \cancel{f_{3,1}} - \cancel{f_{4,1}} = q_1$

$i=2$ $-\cancel{f_{12}} - f_{3,2} - f_{4,2} + \cancel{f_{2,1}} + f_{2,3} + f_{2,4} = q_2$

$i=3$ $-\cancel{f_{1,3}} - f_{2,3} - \underline{\underline{f_{4,3}}} + \cancel{f_{3,1}} + f_{3,2} + \underline{f_{3,4}} = q_3$

$i=4$ $-\cancel{f_{1,4}} - f_{2,4} - \underline{f_{3,4}} + \cancel{f_{4,1}} + f_{4,2} + \underline{\underline{f_{4,3}}} = q_4$

Hence, if the problem is feasible, $\sum_{i\in V} q_i = 0$.

The set V can be partitioned into two subsets

$S, \quad \bar{S} = V - S$

$S = \{ i \mid q_i > 0 \}$ flow enters $\quad \bar{S} = \{ i \mid q_i \leqq 0 \}$

$\bar{S}$ Can be further partitioned into

$T, \quad \bar{S} - T = \bar{\bar{T}}$

$T : \{ i \mid q_i < 0 \}$, flow output $\quad \bar{\bar{T}} : \{ i : q_i = 0 \}$

Now we show how to make a new problem (equivalent to the old) with $q_i^{new} = 0 \;\; \forall i$

We add two extra nodes s, t,

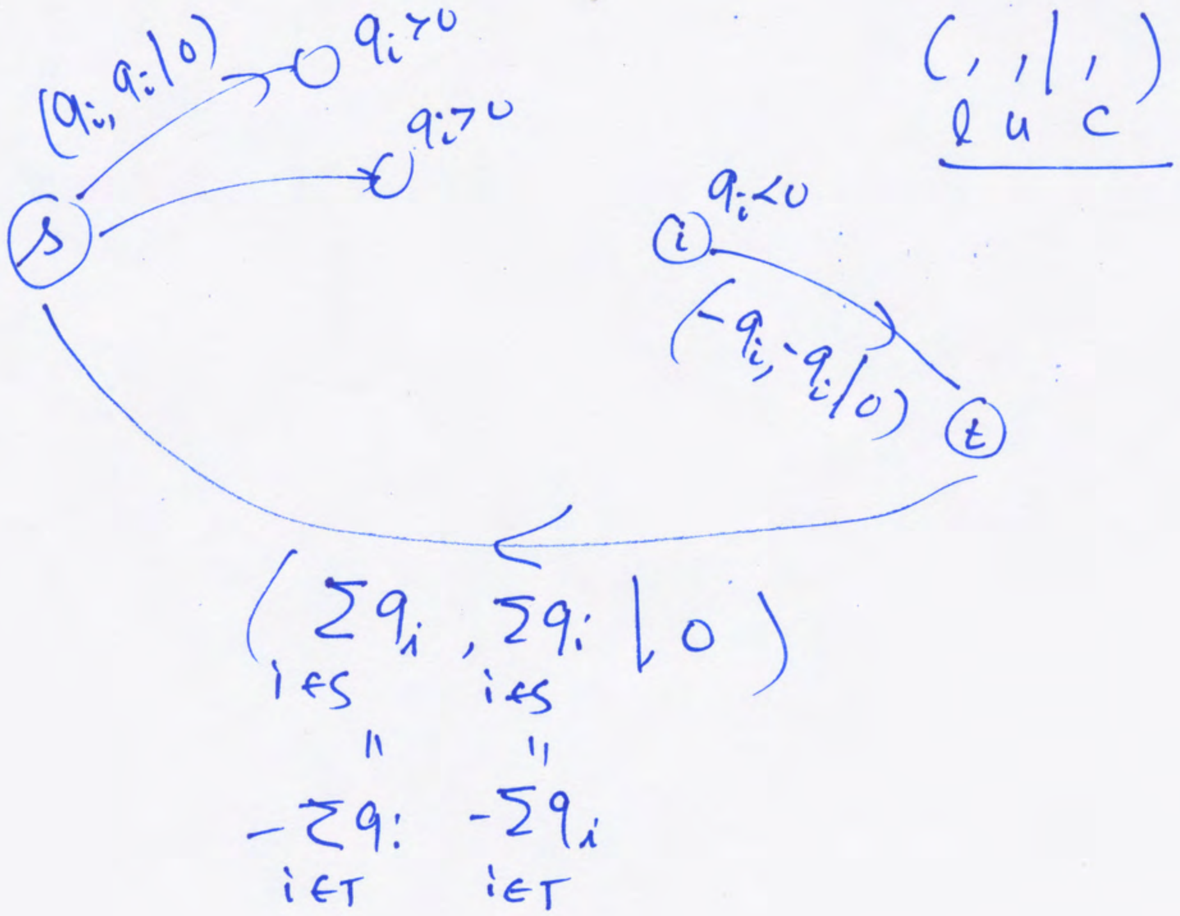
add: Edges: $(s,i): i \in S$; $l_{s,i} = q_i$ $i \in S$
$= u_{s,i}$

$(i,t): i \in T$

$l_{i,t} = u_{i,t} = -q_i$ $i \in T$

edge (t,s): $l_{t,s} = u_{t,s} = \sum_{i \in S} q_i = \sum_{i \in T} -q_i > 0$.

Cost of all these edges $= 0$.



Thus, our min-cost flow problem can be assumed to be: (this is w/o loss of generality)

$$l_{ij} \leq f_{ij} \leq u_{ij} \quad \forall (i,j) \in E$$

$$\sum_j f_{ij} - \sum_j f_{ji} = 0 \quad \forall i \in V \; **$$

$$\text{Min} \sum_{(i,j)\in E} c_{ij} f_{ij}$$

This is known as the "Circulation" formulation. And is convenient to use for some algorithms.

If we only had (**), feasible set would be a vector space (preserved under addition and <u>scalar</u> multiplication)

Edmonds-Karp Scaling on out-of-kilter Method

Bounds - Scaling of Edmonds Karp

Here we assume that l_{ij}, u_{ij} are integers. Let p be the smallest positive integer such that

$$2^p \geq \max_{(i,j)} \{ |u_{ij}|, |l_{ij}| \}$$

Let $u_{ij}^k = \left\lceil \frac{u_{i,j}}{2^{p-k}} \right\rceil$; $l_{ij}^k = \left\lfloor \frac{l_{ij}}{2^{p-k}} \right\rfloor$ $\quad k = 0, 1, \cdots, p$

l_{ij}^0, u_{ij}^0 are 0/1.

<u>Note</u>: $2^{p-k} \cdot u_{ij}^k \geq u_{ij}$; $2^{p-k} l_{ij}^k \leq l_{ij}$ $\quad \forall k.$

For $k = p$, $u_{ij}^p = u_{ij}$; $l_{ij}^p = l_{ij}$

Moreover;

$$\left.\begin{aligned} 1 + \frac{u_{ij}}{2^{p-k}} &> u_{ij}^k \geq \frac{u_{ij}}{2^{p-k}} \\ \text{\& } \quad 1 + \frac{u_{ij}}{2^{p-k-1}} &> u_{ij}^{k+1} \geq \frac{u_{ij}}{2^{p-k-1}} \end{aligned}\right\} **$$

Combining (**) we get

$$-2 < u_{ij}^{k+1} - 2u_{ij}^k < 1$$

Which in turn implies

$$-1 \le u_{ij}^{k+1} - 2u_{ij}^{k} \le 0$$

Similarly, from

$$\frac{l_{ij}}{2^{p-k}} - 1 < l_{ij}^{k} \le \frac{l_{ij}}{2^{p-k}}$$

$$\& \quad \frac{l_{ij}}{2^{p-k-1}} - 1 < l_{ij}^{k+1} \le \frac{l_{ij}}{2^{p-k-1}}$$

We get $\quad -1 < l_{ij}^{k+1} - 2l_{ij}^{k} < 2$

which implies

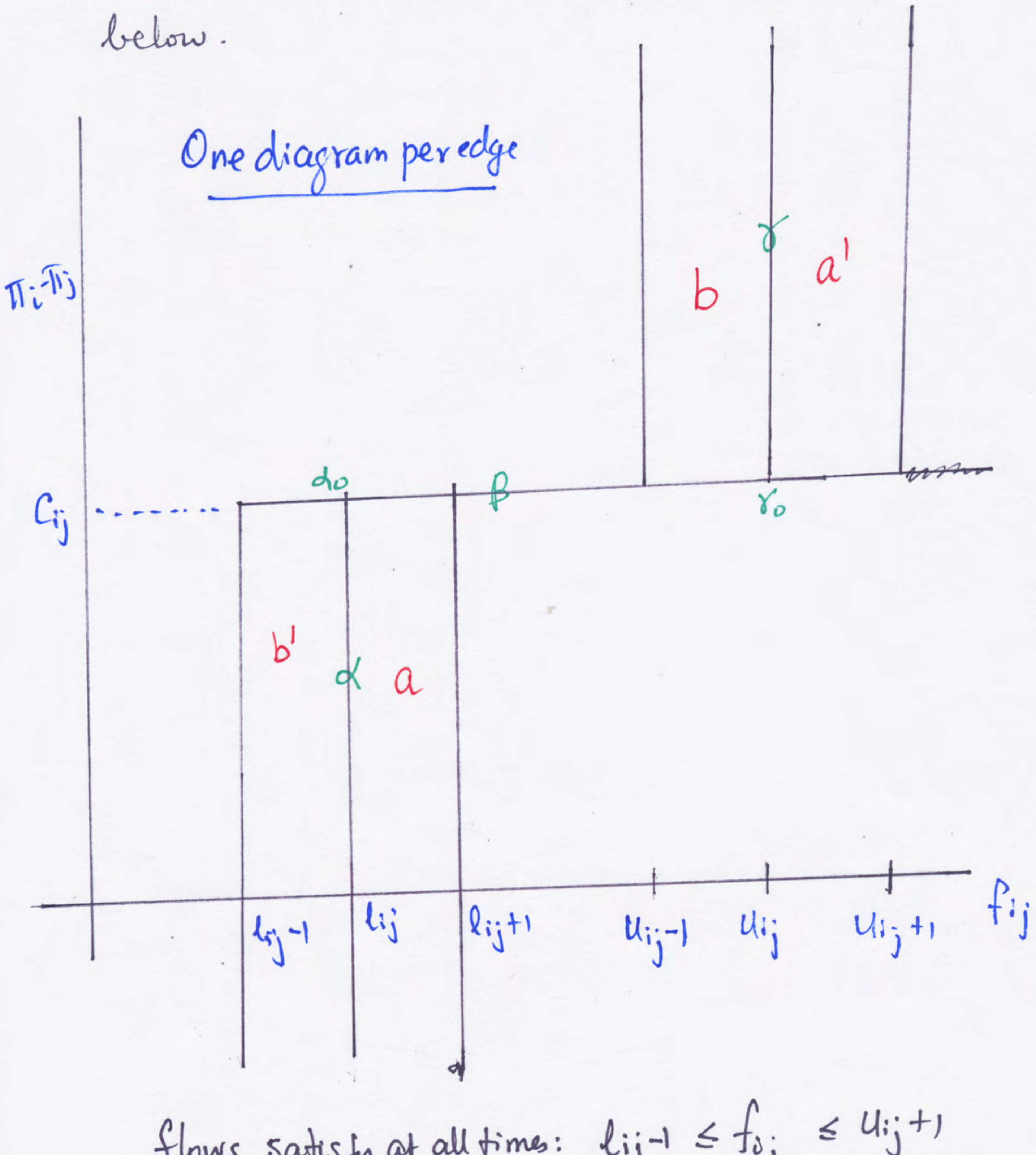
$$0 \le l_{ij}^{k+1} - 2l_{ij}^{k} \le 1$$

Algorithm starts with l_{ij}^{0}, u_{ij}^{0} and $f_{ij}^{0} = 0 \ \forall (i,j) \in E$
and arbitrary π^0 (**)

At k^{th} major cycle, we solve the problem with l_{ij}^{k}, u_{ij}^{k} starting from flow $2(f^*)^{k-1}$ (optimal flow for previous problem and $\pi = \pi^{*(k-1)}$)

We maintain certain conditions throughout the algorithm. This is explained in the next page.

In this process, $\{f, \pi\}$ will always satisfy the following (7)
Condition which are incorporated in the diagram
below.



flows satisfy at all times: $l_{ij}-1 \le f_{ij} \le u_{ij}+1$

f_{ij}, l_{ij}, u_{ij} integers

We now have two more states: a', b' which violate
Original bounds.

However, bounds are violated by <u>1 unit</u> only.

We have <u>also</u> restricted the regions b and a so that these two are only one unit away in terms of CS flows.

Permitted flow changes and dual variable changes for States that were there before, are same as before
$\alpha, \alpha_0, \beta, \gamma, \gamma_0, \underbrace{a, b}$ → flow change of 1 unit only

For new states:

Similar to $\underline{a, b}$ $\begin{cases} b' : \text{flow increase is permitted (1 unit only)} \\ a' \quad \text{" decrease " (1, unit only)} \end{cases}$

b': $(\pi_i - \pi_j)$ permitted to increase within limits ~~decrease without limits~~

a': $(\pi_i - \pi_j)$ " " decrease " "

If we assume l_{ij}, u_{ij} are integral and hence f_{ij} are also integral, <u>and</u> if our starting solution has for each edge in States $\{\alpha, \alpha_0, \beta, \gamma, \gamma_0, a, b, a', b'\}$

Any time flow change subroutine is successful, in one such path, an out-of-kilter edges goes in kilter. We know this must happen in poly. # of dual variable changes without flow change.

Hence alg. will be <u>Strongly Polynomial</u>
for ~~each~~ reducing # of out-of-kilter edges

We start the algorithm with the problem $\{l^0_{ij}, u^0_{ij}, c_{ij}\}$. [Recall for this algorithm, we assume, without loss of generality, that $q_i = 0 \; \forall i \in V$]

Starting flow $f_{ij} = 0 \; \forall (ij) \in E$; arbitrary π^0.

$l^0_{ij}, u^0_{ij} \in \{0, 1\}$ ← Note

Hence at the start, all edges are in one of $\{\alpha, \alpha_0, \beta, \gamma, \gamma_0, a, b, a', b'\}$.

Also since $2^{p-k} u^k_{ij} \geq u_{ij}$; $2^{p-k} l^k_{ij} \leq l_{ij}$ $\forall (ij) \in E$ and $\forall k$

if the original problem is feasible, so are the starting problem and every succeeding problem. When $k = p$, we get the original problem.

$$(f^0, \pi^0) \xrightarrow{O.K.A} \{f^{*}_0, \pi^{*}_0\}$$

↓

SP time.

$$\{f^{*k}, \pi^{*k}\} \rightarrow \{2f^{*k}, \pi^{*k}\} \longrightarrow \{f^{*(k+1)}, \pi^{*(k+1)}\}$$

until $k = p$.

starting solution for $(k+1)^{st}$ problem

If $l_{ij}^{(k)} \leq f_{ij}^{*k} \leq u_{ij}^{k} \quad \forall (ij) \in E$

$$\Downarrow$$

$$l_{ij}^{(k+1)} - 1 \leq 2 l_{ij}^{(k)} \leq 2(f_{ij}^{*})^{k} < 2 u_{ij}^{k} \leq u_{ij}^{(k+1)} + 1$$

Hence $l_{ij}^{(k+1)} \leq 2(f_{ij}^{*})^{k} \leq u_{ij}^{(k+1)}$

Hence these edges will continue to be in-kilter

But those at the bounds may be off by 1 unit

of flow. $\therefore \{2(f_{ij}^{*})^{k}, (\Pi_i^{*})^{k}\}$ Satisfy the conditions

required to start the next phase.

Complexity: p phases
each is SP. $\therefore$ alg (weakly) polynomial in # of bits in input

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There is another algorithm (Similar in spirit) that ~~Sc~~ Scales c_{ij} due to H. Rock

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