CS6381: Lecture #16 10.1.2020

In general, we have $G_S \to$ Shrunken graph

G_S^{eq} : Shrunken equality graph

M_S : " matching in G_S^{eq}

T_S : " tree w.r.t. to M_S in G_S^{eq}.

Nodes of G_S, G_S^{eq}, T_S may be either (Original) real nodes of G or pseudo-nodes and for T_S: we have four possibilities

(i) real-outer-node

(ii) pseudo-outer-node

(iii) real-inner-node

(iv) pseudo-innernode.

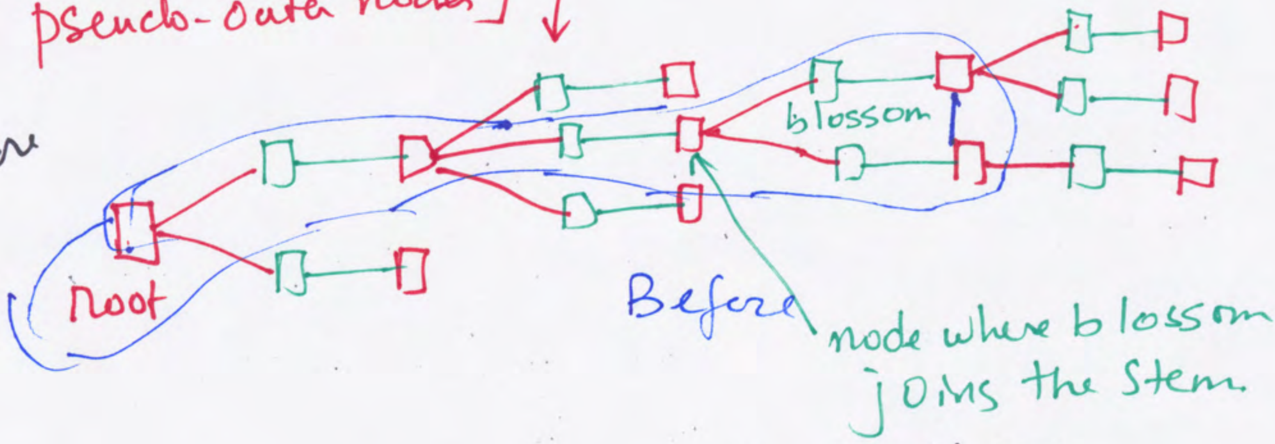
# of real nodes inside any pseudo-node is always odd starting from when they are first Created. We may have pseudo-nodes inside pseudonodes. We remember the <u>odd cycle</u> that created each "layer" for pseudo-nodes

The term "Current" pseudo-node refers to the Outermost (equivalently the largest set of the nested family for a pseudonode.

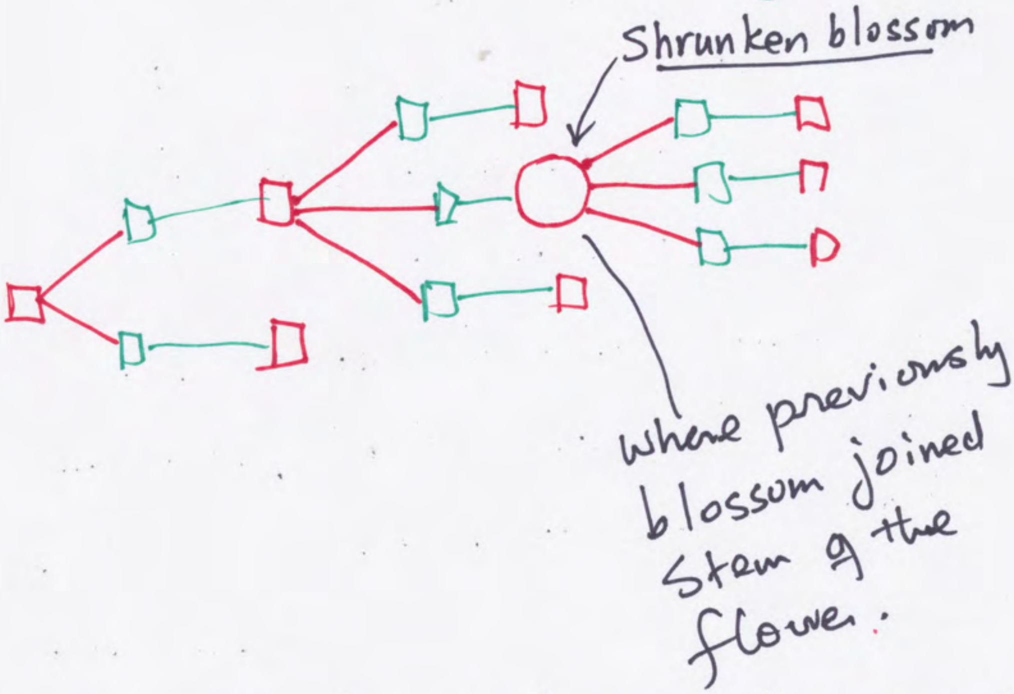
Pseudo nodes inside pseudo-nodes are <u>not</u> <u>current</u>.

Since pseudo-nodes are <u>not</u> <u>unshrunk</u> unless we are <u>forced</u> to do so, there could be pseudo-inner nodes [When they are first formed, they will be pseudo-outer nodes]

before



after



There are only two instances where we are forced to unshrink (the outer most layer) of a pseudonode: (i) When we have a PM in shrunken equality graph; (ii) $y(S) \downarrow 0$. for a pseudo inner node

First case we have a PM not only in Shrunken equality graph but from that we get a PM in equality graph & hence optimal solution to P3.

If at some stage, we are unable to augment the matching in the current equality shrunken graph, or grow the tree, or shrink a blossom, then outer nodes of the tree (real or pseudo) form a Hungarian set in the Shrunken equality graph G_S^{eq}. Now we change dual variables. If the set is also a Hungarian set in the Shrunken graph G_S, then we show G does not have a Perfect Matching and the dual problem is unbounded and we stop. Else, we change dual variables as shown in the next page.

$$y_i^{new} = \begin{cases} y_i^{old} - \epsilon & \text{if } i \text{ is real outer node of } T_s \text{ or inside a pseudo-outer-node of } T_s \\ y_i^{old} + \epsilon & \text{if } i \text{ is real inner node of } T_s \text{ or inside a pseudo-inner node of } T_s \\ y_i^{old} & \text{else.} \end{cases}$$

$$y^{new}(S) = \begin{cases} y^{old}(S) + 2\epsilon & S \text{ is a "current" pseudo-outer-node of } T_s \\ y^{old}(S) - 2\epsilon & S \text{ is a "current" pseudo inner-node of } T_s \\ y^{old}(S) & \text{else.} \end{cases}$$

where $\epsilon \geq 0$ is chosen as large as possible subject to the following conditions

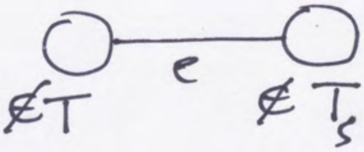
I. For every edge e in $G_s - G_s^{eq}$ whose one end is an outer node of T_s and the other is not in T_s, $\epsilon \leq \underbrace{f_e(y) - c_e}_{>0}$

II. For every edge e of $G_s - G_s^{eq}$ both of whose ends are outer nodes of T, $\epsilon \leq \underbrace{\frac{f_e(y) - c_e}{2}}_{>0}$

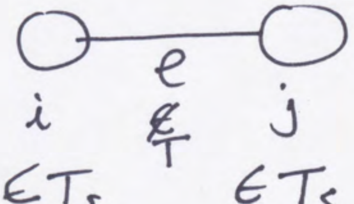
III For every S representing a current pseudo-inner node of T, $\epsilon \leq \frac{y(S)}{2}$ ← this could make $\epsilon = 0$

If ~~$\epsilon = 0$~~ this is case → S is unshrunk.

Effect of dual variable changes on various structures introduced so far:

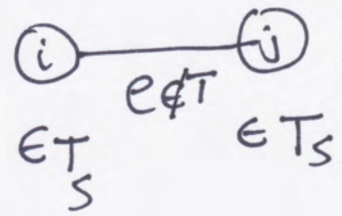
1. 

$f_e(y)$ is <u>not</u> changed.

2. 

(i,j) inside same pseudo-node (whether it is inner or outer)

$f_e(y)$ is not changed

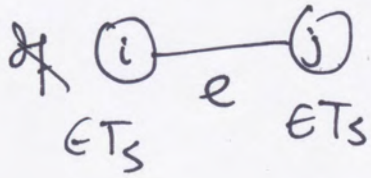
3. 

: $\left.\begin{matrix} i \\ j \end{matrix}\right\} \in$ outer node or inside a pseudo-outer node of T_s but different pseudo-nodes

$f_e(y)$ decreases by 2ϵ.

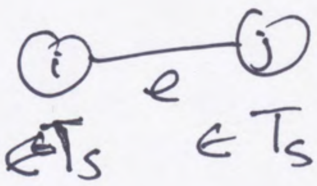
This is the reason for II to limit ϵ.

(This has potential to shrink further)

4. 

$i \in$ inner ; $j \in$ outer
or or
$=$ $j =$

$f_e(y)$ unchanged.

5. 

$\left.\begin{matrix} i = \text{ or } \in \text{ inner} \\ j = \text{ or } \in \text{ inner} \end{matrix}\right\}$ e may not be in new $G^{eq}_{.}$, G^{eq}_{s}

But such an edge does not matter. Such an edge $\notin M_s$, or T_s.

$f_e(y)$ increases.

6. $(i) \overset{e}{\text{———}} (j)$: may drop out from G_S^{eq}, G^{eq}, but not in M_S, T_S : no problem

i: inner; $j \notin T_S$

<u>$f_e(y)$ increases</u>

7. $(i) \overset{e \in G_S}{\text{———}} (j)$: $f_e(y)$ decreases

i: Outer; $j \notin T_S$

but (I) limits decrease

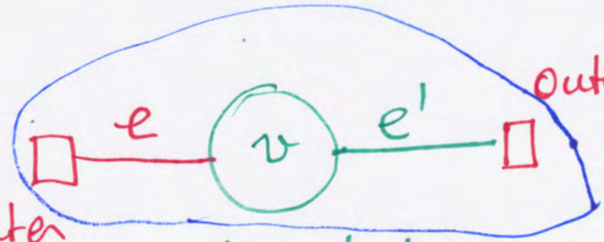
So new y is feasible

$\epsilon \le f_e(y) - c_e$.

If ϵ is determined by (I), T_S will grow. (M_S might or might not grow in size)

While changing the dual variables, if ϵ is determined by III, then $y^{new}(S) = 0$ for S that was limiting ϵ. for some <u>pseudo-inner</u> node of T_S. This is one instance where we are "forced" to unshrink S. <u>However, this is only done for the "outermost" layer in S.</u> ie We do not unshrink any pseudo node contained (proper) in S. Outer layer of the "Onion" is peeled.

The unshrinking process is shown below.



Outer node (real or pseudo)

outer node (real or pseudo

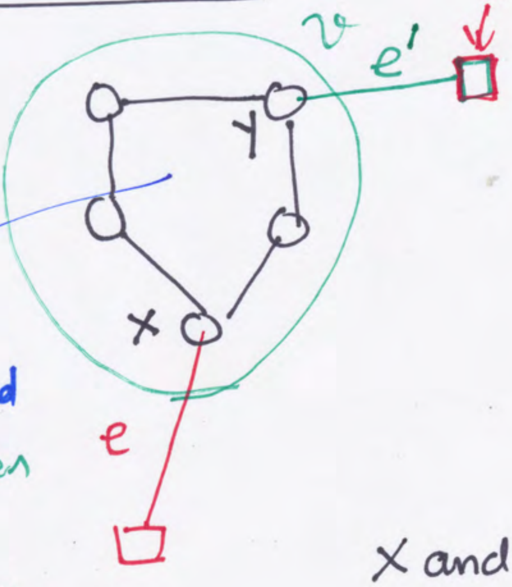
pseudo-inner node S of T_s determining e in III

→ Part of T_s that include S.

Shrunken

pseudo-inner node of T_s

Expanded View



outer (real)

odd cycle that created pseudo inner node v prior to this step.

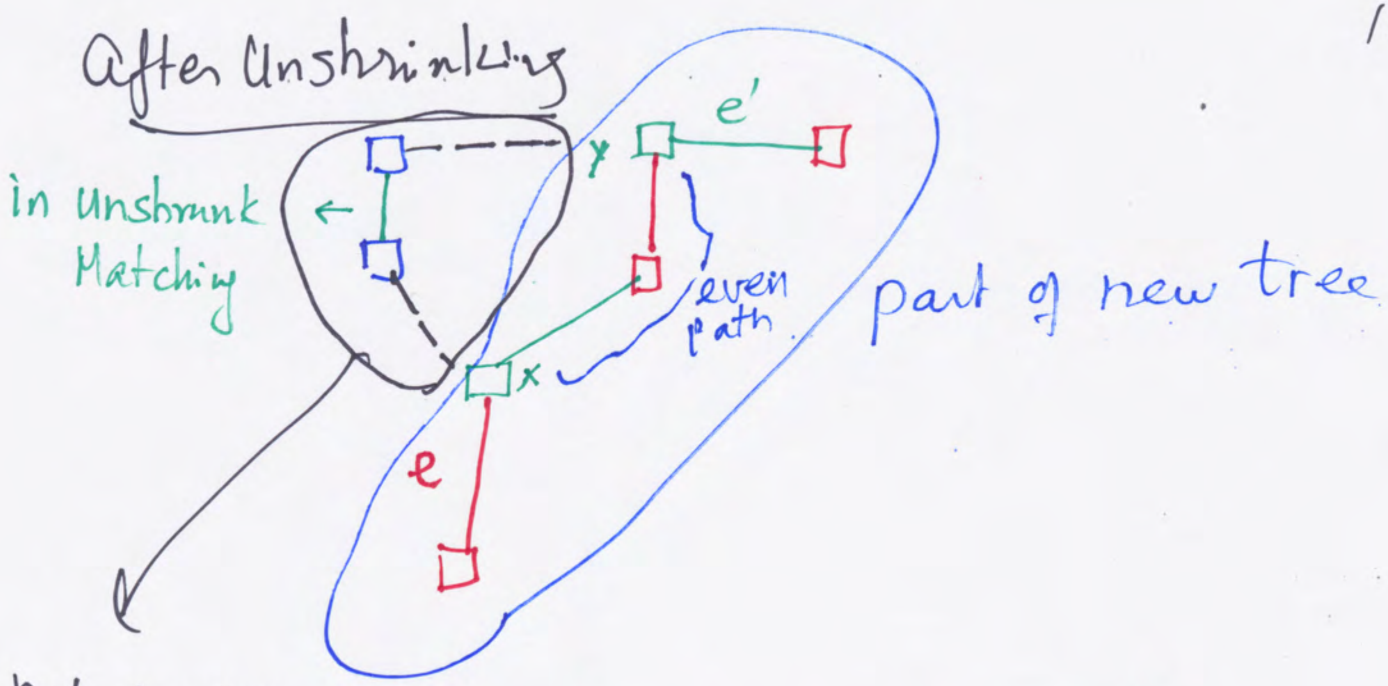
Before Unshrinking

x, y are nodes inside pseudo inner node that are parts of edges e → x

e' → y

x and y split the cycle into 2 parts (paths); Since cycle has odd number of edges, One of these is odd and the other is even.

After Unshrinking



Not in new Tree. , But in G_S^{eq}, Some new edges in new M_S.

Please note: only pseudo-inner nodes are unshrunk while same tree (or its unshrunk version) is retained. Pseudo-outer nodes of a tree are <u>NOT</u> unshrunk as long as matching is not grown in size.

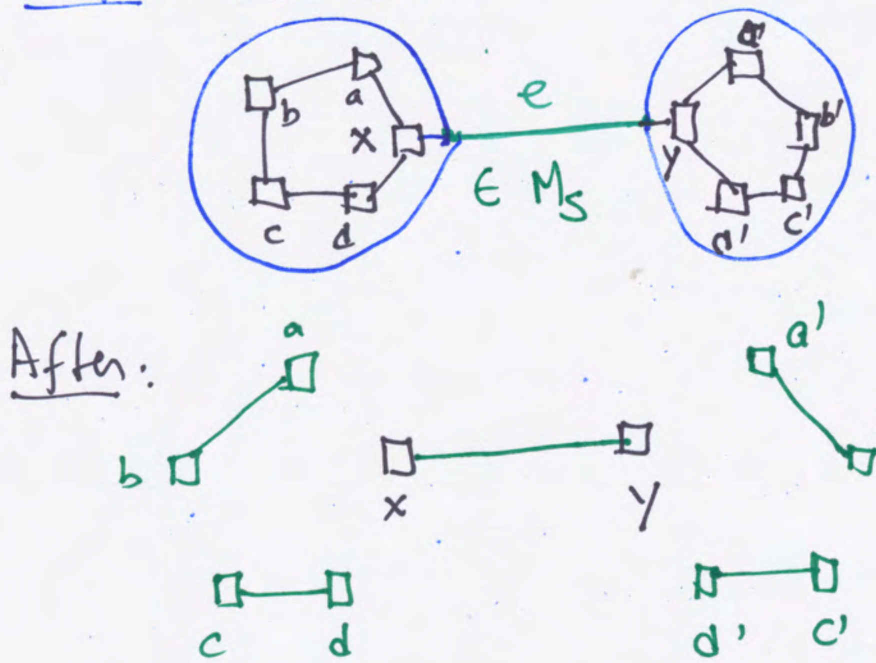
After this we try to go back to R1 : increase Tree size, new shrinkings, increase Matching. etc.

Termination Conditions are discussed in the next page.

Termination of the algorithm

Ⓐ If we find a perfect matching in G_S^{eq} at some step, we get a PM in G^{eq} by unshrinking one layer at a time of pseudo-nodes as shown below.

Before Unshrinking



Recall edges of the odd cycle that created pseudo nodes are always in G^{eq}

Ⓑ. The other possible termination occurs when when at some stage of the algorithm we are unable to increase matching size, augment the tree, or shrink. At this point set of outer nodes of T_s is a Hungarian set in G_S^{eq} and also in G_s.

Further, there are no pseudo-inner nodes to limit ϵ. Thus, $\epsilon = \infty$ in this case, and the dual is unbounded. This, using LP, implies that Primal LP is infeasible and hence has no integer feasible solution as well. Thus, although we ~~do not have a~~ it is not guaranteed that there is a Hungarian Set in G, there is no perfect matching in G, because out of each pseudonode, at least one real node inside will have to matched to a real node outside this set (since size is odd) and there are not enough to go around.

<u>Complexity</u>: # of times $|M|$ can increase is $\left\lfloor \frac{|V|}{2} \right\rfloor$.

# of times we shrink without $|M|$ increasing, is $\leq \frac{|V|}{2}$, → this results in creating a pseudo-outer-node.

Unshrinking is on pseudo-inner-nodes $\therefore \leq |V|/2$ times.

→ next page

Thus, in (strongly) polynomial time we must either find a Hungarian Set in G_S (indicating no PM exists in G) or augment the matching. Thus, our algorithm is strongly polynomial and is valid.

In the next lecture we turn to some applications:

1) SP in undir. graphs with no negative (undir) cycles (J. Edmonds) Promise # 3 in CS6363
2) (Chinese) Postman Problem [Guan Mei-Go]
3) 2-machine UET Shop with precedence Constraints : Fujii, Kasami, Ninomiya
4) Bi-directed Flows → b-matching
5) T-joins & T-cuts → Multicommodity flows