Bipartite Flows & b-matching

Consider the problem:

\[ \text{x} \geq 0, \text{ integer}, \]

\[ Ax = b \text{; A integer matrix} \]

\[ \text{A} \geq 0, \text{ \pm 1 matrix} \]

\[ b \text{ integer vector} \]

\[ \min(\max) \text{ c}_x \text{; c arbitrary} \]

When \( a_{i,j} \) (entries in A) satisfy the condition:

\[ \sum_{i} \left| a_{i,j} \right| \leq 2 \forall j. \]

(Each column of A contains either 2 or less \( \pm 1 \), or one \( \pm 2 \), or none of these)

Such a problem is called bipartite flows.

If each column has either one non-zero which is \( \pm 1 \), or one \( +1 \) and one \( -1 \), we get our (directed) min-cost flow problem.

If it has two \( +1 \) or two \( -1 \), we get b-matching. \( \pm 2 \) allows SP in undir. graphs.
Algorithms for matching can be extended to polynomial algorithms for these as well (and also can be converted to SP-algo strongly polynomial algorithm).

**Clique in Special graphs**

Given an undirected graph $G = [V; E]$ with edge weights $w(e), e \in E$.

**Def** Line graph $L(G)$ of $G$ is defined as follows. Let there are vertices in $L(G)$.

$L(G) = \left[ E ; F \right]$

$F = \{ (e, e') : e, e' \in E, e = e' \text{ share a vertex in common such that} \text{ e is incident at some vertex} \}$

**Example**

![Diagram](image)
A matching \( M \) in \( G \) is an anti-clique (independent set) in \( L(G) \) and conversely. Find max cardinality matching in \( G \) is the same as finding largest independent set in \( L(G) \) and similar results hold for max weighted matching and max wt cl ind. set.

Notice that \( L(G) \) can not have as a node-induced subgraph, a claw.

Such graphs are called Claw-free graphs. Not all claw-free graphs are line graphs of some graph. So finding largest (wtd) ind. set in claw-free graphs extends match.
There are several other applications of matching (see book by L. Lovasz and M. Plummer).

One of these applies to multi-commodity flows and goes under the name T-joins and T-cuts. Another deals with decomposing a polygon into convex polygons (see Chazelle and Dobkin).

There is also an application in physics on spin models.

End of Matching

We move next to multi-commodity flows. Download A. Schrijver's notes.