Multi-Commodity Flows

The mathematical reason for considering this, is NOT that apples and oranges are being sent. It is that origin-destination pairs are different. Most important applications are from Telecommunications area. Message flow from $A \to B$ and $C \to D$ should be <u>separately</u> conserved. In general, there may also be individual "commodity" capacities on edges in addition to "overall" capacities.

The most general problem may be of the form

$$l_{ij}^{(k)} \le f_{i,j}^{(k)} \le u_{ij}^{(k)} \qquad (i,j) \in E, \quad k = 1 \dots K$$

$$l_{ij} \le \sum_k f_{ij}^{(k)} \le u_{ij} \qquad (i,j) \in E$$

$$\sum_j f_{ij}^{(k)} - \sum f_{ij}^{(k)} = \begin{cases} F_k & i = s_k \\ 0 & i \ne s_k, t_k \\ -F_k & i = t_k \end{cases}$$

Where (s_k, t_k) are origin-destination for commodity k.

$$\text{Max } \sum_{k=1}^{K} P_k F_k - \sum_{(i,j) \in E,\, k} C_{ij}^k f_{ij}^k$$

This is a ~~f~~ linear program and can (in theory) be solved in strongly polynomial alg time by work of E. Tardos & A. Frank. However, these are __not__ __Combinatorial algorithms__.

The earliest work is due to W.S. Jewell and FF in 1958 as far as we know.

In the undirected case for maxflow, we have $f_{ij}^{(k)}$: +, 0, -

$$\sum_j f_{ij}^{(k)} = \begin{cases} F_k & i = s_k \\ 0 & i \neq s_k, t_k \\ -F_k & i = t_k \end{cases}$$

$$0 \le |f_{ij}^{(k)}| \le u_{ij}^{(k)} \; ; \qquad (ij) \in E, \; k = 1 \ldots K.$$

$$0 \le \sum_k |f_{ij}^{k}| \le u_{ij} \qquad (ij) \in E$$

$$\text{Max} \sum_k |F_k|$$

We want first to consider maximum flow problem only. We attempt to replicate as many of the results in $k=1$ case, for this problem in __undir.__ graphs. Fewer results for __directed__ version.

See also Notes by A. Schrijver and the paper by (3)
M. Lomonosov in Discrete Applied Math & The book
by A. Frank. (on disjoint paths). The three volume
book on Combinatorial optimization by A. Schrijver
is a very good reference book. If you can
read (and understand) Russian, there is a book
edited by A. Kelmans (I think) That has much of
the Russian Work until that time.

There are four main points regarding single
Commodity max flow problem (One of which
is never stressed while doing single Commodity
Version That needs to be considered here)

1. Max-flow-min-cut theorem (duality)
2. Existence of a Strongly polynomial Combinatorial algorithm.
3. Integrality property: If data is integral
   $\exists$ integral (f, F) That are optimal to LP.
4. Given $G = [V; E]$ with edge capacities
   $u_e : e \in E$, and a specific value F^0 of
   the total flow, Can this be achieved?

Feasibility Question

The last question was <u>not</u> stressed in Single Commodity case. Because, if F^* is max flow, F^0 is feasible iff $F^0 \leq F^*$ (for problems with 0 lower bounds, and no external flows except at s and t) [If lower bounds are not 0, then we can find lower bound on $\underline{F}^*$ and upper bound on $\overline{F}^*$ and F^0 is feasible iff $\underline{F}^* \leq F^0 \leq \overline{F}^*$]

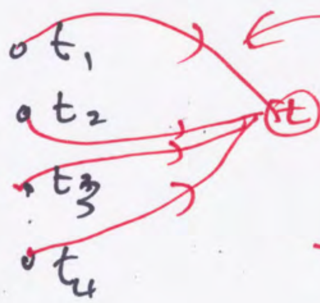
But this question is important for multi-commodity problems.

Before we continue, let us discuss those multi-com. flows problems that can be <u>converted</u> to single commodity problems.

<u>For directed Version</u> (undir?): if $s_k = s \;\forall k$, (i.e $\exists$ a single origin) or if $t_k = t \;\forall k$ (i.e $\exists$ a single destination), We can do this

Single dest? case done similarly

(3)



∞ cap if max flow

↳ Specified Values if min cost

Cost = 0

So we will assume from now on, that this is NOT the case.

The first such case was solved by T.C. Hu (196x) and improved by M. Sakarovitch and also by A. Whinston & B. Rothschild. (a good description is in the notes (downloadable(?)) by A. Schrijver.

This is the case with $k=2$, all of $\{s_1, s_2, t_1, t_2\}$ distinct. with an undir. graph for max-flow and capacities on edges are for the sum of flows of both commodities (common upper bound; no individual comm. bounds)

(This is what is done in most of the literature)

$$\left| f_{ij}^{(1)} \right| + \left| f_{ij}^{(2)} \right| \le u_{ij} \quad \forall (i,j) \in E.$$

equivalent to

$$f_{ij}^{(1)} + f_{ij}^{(2)} \le u_{ij}$$
$$f_{ij}^{(1)} - f_{ij}^{(2)} \le u_{ij}$$
$$-f_{ij}^{(1)} + f_{ij}^{(2)} \le u_{ij}$$
$$-f_{ij}^{(1)} - f_{ij}^{(2)} \le u_{ij}$$

$f_{ij}^{(k)}$ allowed to be <u>unrestricted</u> in sign.

$$\sum_j f_{ij}^{(k)} = \begin{cases} F_k & i = s_k \\ 0 & i \neq s_k, t_k \\ -F_k & i = t_k \end{cases} \qquad k = 1, 2$$

(flow conservation <u>separately</u> for each comm.)

<u>Def</u>: A (joint) <u>disconnecting</u> <u>set</u> (cut) is a set of edges whose removal leaves no path between s_k and t_k $\forall k$.

<u>Def</u> ~~the~~ Let $c(s_1, s_2, \ldots s_k ; t_1, \ldots t_k)$ represent min-cut that separates s_i from t_i $i = 1 \cdots k$.

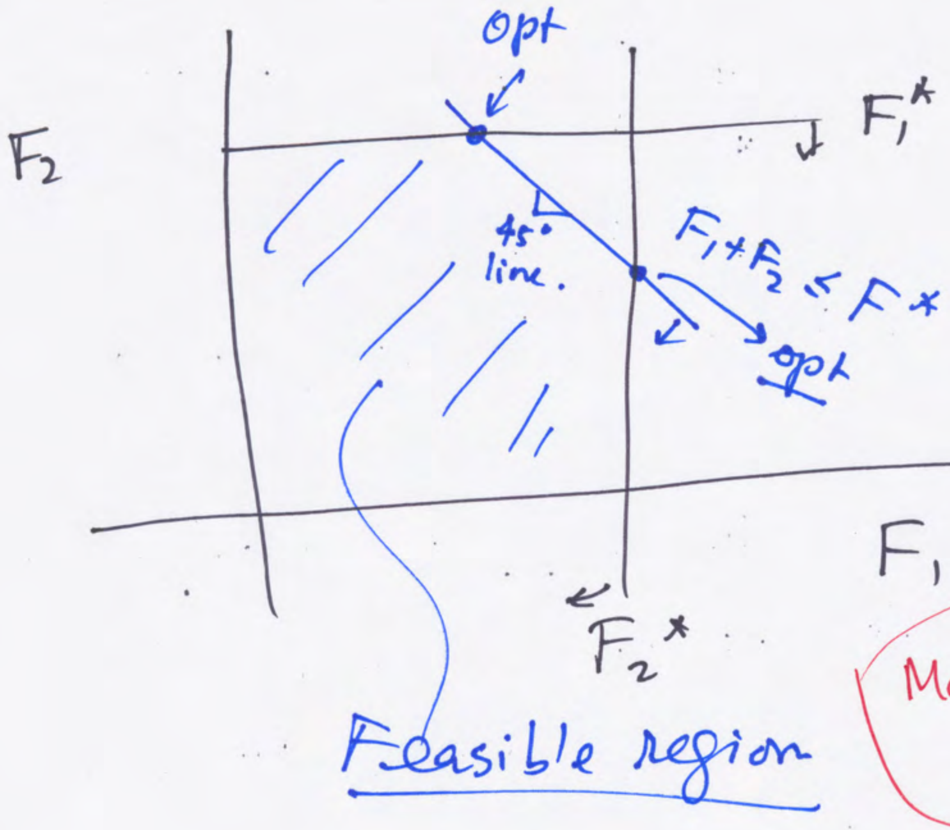
<u>Lemma 1</u>
$$c(s_1, s_2 ; t_1, t_2) = \min \Big\{ c(\underbrace{\{s_1, s_2\}}_{\text{condensed}}, \underbrace{\{t_1, t_2\}}_{\text{condensed}}) ; \; c(\underbrace{\{s_1, t_2\}}_{\text{condensed}}, \underbrace{\{s_2, t_1\}}_{\text{condensed}}) \Big\}$$

<u>Lemma 2</u>: (F_1, F_2) is feasible iff

$$\left. \begin{array}{l} F_1 \leq c(s_1, t_1) \\ F_2 \leq c(s_2, t_2) \end{array} \right\} \text{Single comm.}$$

$$F_1 + F_2 \leq c(s_1, s_2 : t_1, t_2)$$

(Main result of T. C. Hu.



Max-Flow-Min-Cut generalized

Proof: That $F_1 \leq C(s_1; t_1) = F_1^*$

$F_2 \leq C(s_2; t_2) = F_2^*$

needs no further explanation. It follows from single commodity max flow results.

That $F_1 + F_2 \leq C(s_1, s_2; t_1 t_2)$ also needs no explanation since this also follows from max-flow for single commodity. all flows of either commodity must go across such cuts.

What needs proof is that each such (F_1, F_2) is achievable. This is what we do next.

Proof (contd)

~~Let (F_1, F_2) be~~

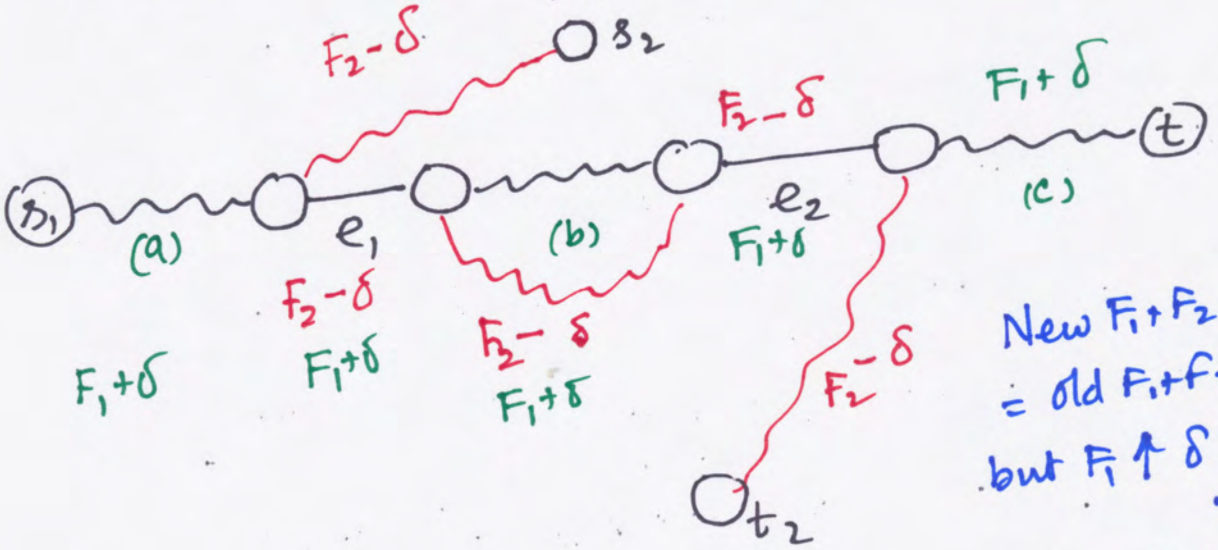
Theorem: $\exists$ an optimal solution (F_1, F_2) to the two commodity flow problem with $F_1 = F_1^*$ (recall F_1^* is max flow of commodity #1 in the absence of second commodity.)

Proof: Let (F_1, F_2) be an optimal solution to the two commodity flow problem ~~in~~ which maximizes F_1. Suppose, in the manner of a proof by contradiction, $F_1 < F_1^*$. This means, that if we ignore the second commodity, there is an augmenting path w.r.t (f', F'). Traversing this path from s_1 to t_1 let e_1, e_2 be the first and last edges that are saturated because of the presence of second commodity flows. i.e both these edges have $|f^2| > 0$.

There are two cases:

i) One path from s_2 to t_2 carrying positive flow uses both e_1 and e_2 (in some direction)

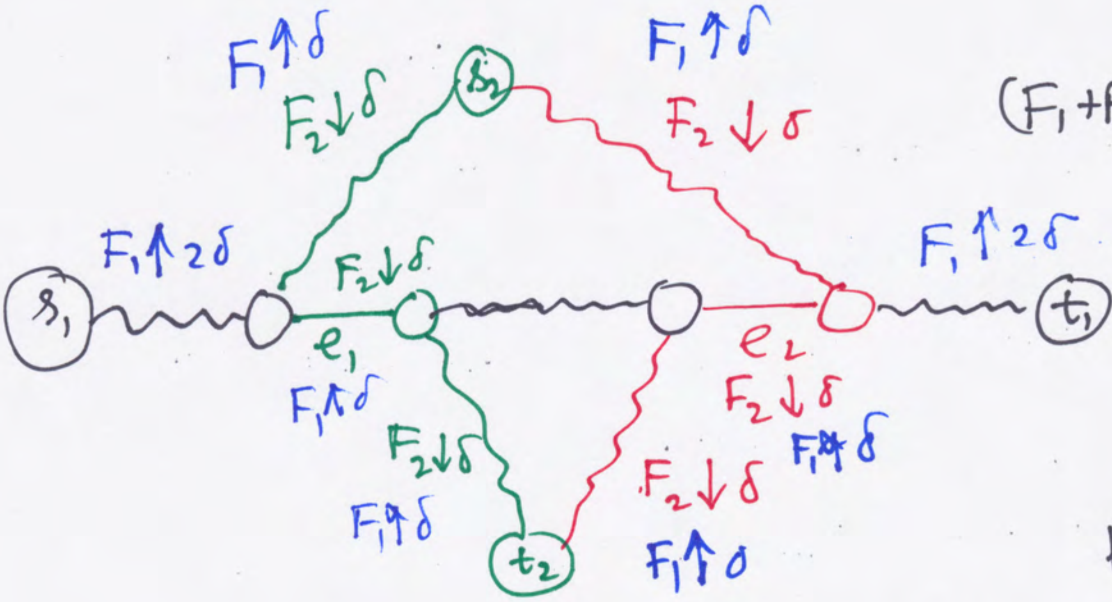
ii) One path from s_2 to t_2 uses e_1 and the other uses e_2. <u>See diagrams below</u> (we will have one on this page and one on next page)



New $F_1 + F_2$ = old $F_1 + F_2$ but $F_1 \uparrow \delta$.

<u>Note</u>: edges in subpath (a) (c) have excess cap. in spite of 2-flows: Since e_1 and e_2 are the first ~~and last edge~~ and last edge that have +ve 2-flow that <u>blocks</u> 1-flow augmentation.

Contradicting F_1 was maximum among optimal 2-com. <u>Solutions</u> in (F_1, F_2).



$(F_1+F_2)^{new} = (F_1+F_2)^{old}$

$F_1 \uparrow 2\delta$

Contradiction to F_1 is max among opt 2-Commodity flows.

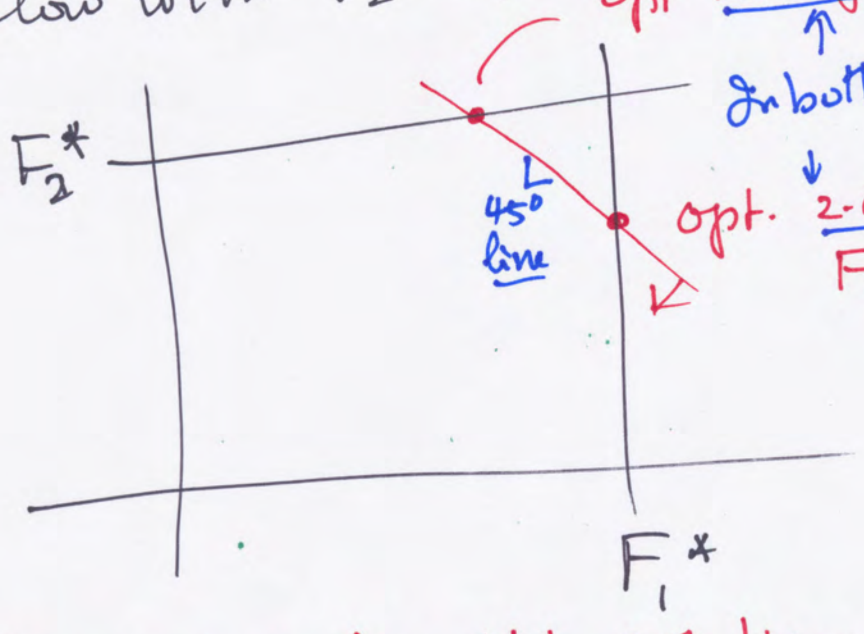
End of Proof

Similar results hold to show $\exists$ an opt. 2-com flow with $F_2 = F_2^*$

opt 2com flow with $F_2 = F_2^*$

In both $F_1 + F_2$ is same

opt. 2-Com. flow with $F_1 = F_1^*$



The set of feasible solutions is convex, Hence the red line is feasible

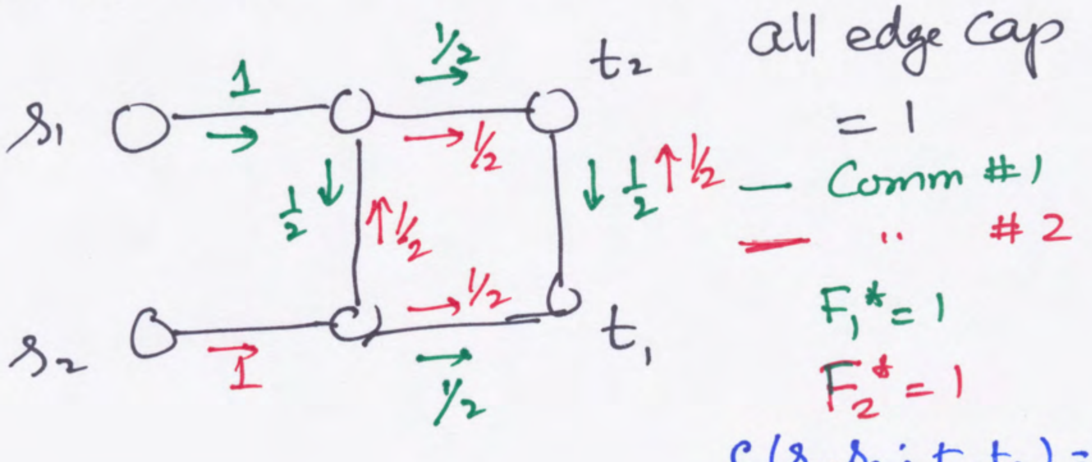
All that remains is to show

$$F^* = c(s_1, s_2; t_1, t_2)$$

And this is done by an algorithm.

T.C. Hu did this via an "augmenting path" algorithm; more precisely an augmenting "double path" algorithm. But he did not show polynomiality. M. Sakarovitch did by a much simpler process. This is what we describe (Here we follow A. Schrijver's notes — available on the internet)

We do this while maintaining half-integrality. We can not avoid ½. Example (P. Seymour)



$c(s_1, s_2; t_1, t_2) = 2$