CS6381: Lecture # 20 10.10.2020 (1)

A. Schrijver's notes: (I am following his notation)

Watchout: Some times origin-destination pairs are referred to as (r_i, s_i) $i = 1 \ldots k$ & Some times as (s_i, t_i) $i = 1 \ldots k$.

k-Commodity flow (directed graph version)

Input: A dir. graph $G = [V; E]$, pairs $(r_1, s_1) (r_2, s_2) \cdots (r_k, s_k)$, a capacity function $c: E \to \mathbb{R}_+$, $\mathbb{Q}_+$, $\mathbb{Z}_+$ and demands $d_1, d_2, \ldots d_k$.

Want: for each $i = 1, \ldots k$, and $r_i \to s_i$ flow $x_i \in \mathbb{R}_+^E$ (ith Commodity.) so that x_i has value (total flow) d_i and so that

$$\sum_{i=1}^{k} x_i(e) \leq c(e) \quad \forall e \in E.$$

(assume $r_i \neq s_i$ for each i)

When we refer to the undir. version, we replace each edge $e = (u, v)$ by two dir. ones $(u \to v)$ and $(v \to u)$. and require

$$\sum_{i=1}^{k} \left(x_i(u,v) + x_i(v,u) \right) \leq c(e) \quad \forall e \in E$$

$$\Updownarrow$$

$$\sum_{i=1}^{k} |x_i(e)| \leq c(e) \qquad \forall e \in E$$

$x_i(e)$ allowed to be +ve, -ve, or 0.

Without duplication of edges.

This is the feasibility problem!

Cut Condition:

$$\sum_{\substack{k:\, s_i \in W \\ t_i \notin W}} d_k \leq \sum_{\substack{e=(i \to j) \\ i \in W \\ j \notin W}} c_e \qquad \forall W \subseteq V.$$

Cut condition is necessary for feasibility:

This is true for both directed and undir. versions of the problem

Proof is easy and left to you all.

It is also sufficient for single commodity problems (and problems that can be converted to single-commodity problems). We will show that the condition modified (for undir version) to read:

$$\sum_{\substack{k:(s_k,t_k)\text{ are}\\ \text{separated by}\\ (W,V-W)}} d_k \le \sum_{\substack{e:\text{ its}\\ \text{ends are}\\ \text{separated}\\ \text{by }(W,V-W)}} c_e \quad \forall\, W \subseteq V$$

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is also sufficient for 2-Comm. undir. version (also called T. C. Hu Problem)

Multi-Commodity problems for which Cut-Condition is sufficient are known as Cut-determined problems. More about these later. (This is also related to integrality of the dual)

We will also give a few examples to show that this condition is NOT sufficient in general

That is done by "Metrics" as in paper by M. IRI et al known in some circles as "The Japanese Theorem".

Feasibility for 2-Commodity flow in undir. graphs

Let $G = [V; E]$ be an undirected graph,
(s_k, t_k) $k = 1, 2$ are "terminals" of the two commodities [assume all are distinct for the most general case].
All lower bounds are 0; there are no capacities for individual commodities separately;
There is only a "joint" capacity $c(e)$ on edge e. $c(e) > 0$ $\forall e \in E$.
Want to ~~maximize~~ satisfy demands d_k, $k = 1, 2$. (assume cut condition holds)

M. Sakarovitch's Work (PhD Thesis 1966, Paper 1973)

Orient edges of G arbitrarily (flow +ve means flow is in the dir. chosen, -ve means the opp. dir)

Let $x \in R^E$ denote any flow (on edges of G)

Define $f(x, v) := \sum_{\substack{e:\ e \text{ is dir.}\\ \text{out of } v}} x(e) - \sum_{\substack{e:\ e \text{ is dir}\\ \text{into } v}} x(e)$ $\forall v \in V$.

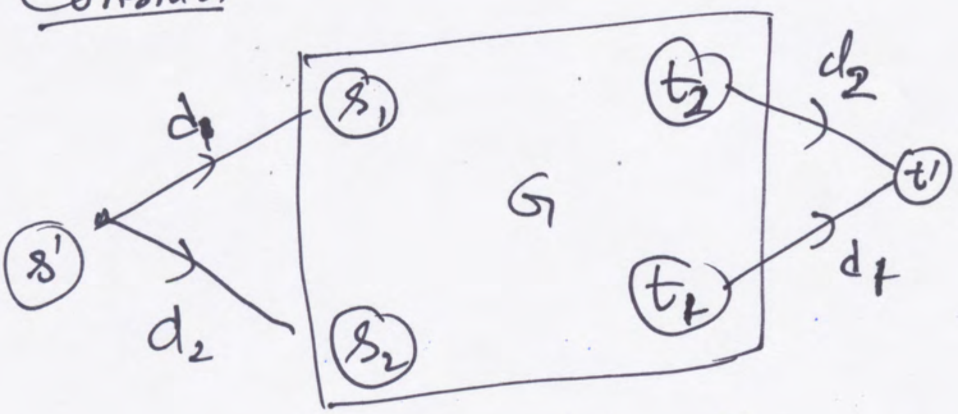
$\downarrow$

net out flow at v.)

Assume $C(e)$ integral for now, as well as d_k

This shows $\frac{1}{2}$ integrality as well.

Consider



Solve max flow (in the single commodity sense between s' and t'.

Since cut condition is satisfied, $\exists$ a max flow $x' \in \mathbb{R}^E$ with value $d_1 + d_2$. Moreover, we may assume that x' is integral since $\{c(e), d_k\}$ are integers.

Here we have

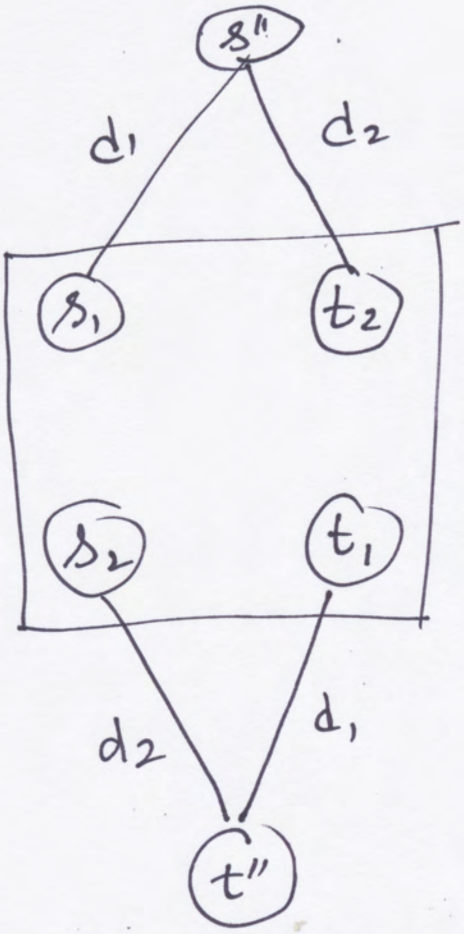
$$f(x', s_1) = d_1, \quad f(x', t_1) = -d_1$$

$$f(x', s_2) = d_2, \quad f(x', t_2) = -d_2$$

$$f(x', v) = 0 \quad \forall\, v \notin \{s_1, s_2, t_1, t_2\}$$

$$|x'(e)| \le c(e) \quad \forall\, e \in E$$

Consider:



Since the cut condition is satisfied, $\exists$ a max flow $x'' \in \mathbb{R}^E$ with value $d_1 + d_2$. Moreover, we may assume that x'' is integral since $\{c(e), d_k\}$ are integral. Here we have

$$f(x'', s_1) = d_1 \ , \ f(x'', t_1) = -d_1$$

$$f(x'', s_2) = -d_2 \ , \ f(x'', t_2) = d_2$$

$$f(x'', v) = 0 \quad v \notin \{s_1, s_2, t_1, t_2\}$$

$$|x''(e)| \le c(e) \quad \forall \ e \in E.$$

Moreover, x', x'' can be found in strongly polynomial time.!

Let $x_1 \in R^E$: $x_1 = \frac{1}{2}(x' + x'')$

$x_2 \in R^E$: $x_2 = \frac{1}{2}(x' - x'')$

Claim that (x_1, x_2) are feasible 2-com. flows.

<u>Proof</u>: Since $f(x, v)$ is linear in x, we have

$$f(x_1, v) = \frac{1}{2}\left[f(x', v) + f(x'', v)\right]$$

$$\therefore f(x_1, v) = 0 \quad \forall v \notin \{s_1, s_2, t_1, t_2\}$$

$$f(x_1, s_1) = d_1, \; f(x_1, t_1) = -d_1$$

$$f(x_1, s_2) = 0 = f(x_1, t_2)$$

& $$f(x_2, v) = \frac{1}{2}\left[f(x', v) - f(x'', v)\right]$$

$$\therefore f(x_2, v) = 0 \quad \forall v \notin \{s_1, s_2, t_1, t_2\}$$

$$f(x_2, s_1) = 0, \; f(x_2, t_1) = 0$$

$$f(x_2, s_2) = d_2, \; f(x_2, t_2) = -d_2$$

~~Hence~~ Hence x_1 is a $1/2$-integer flow from s_1 to t_1

& x_2 is a .. .. .. s_2 to t_2

To show that these also satisfy capacity conditions

We need to show

$$|x_1(e)| + |x_2(e)| = \left|\tfrac{1}{2}[x'(e) + x''(e)]\right| + \left|\tfrac{1}{2}[x'(e) - x''(e)]\right|$$

$$\leq \max\left[\underbrace{|x'(e)|, |x''(e)|] \leq c(e)}_{\text{These follow from how } x', x'' \text{ are constructed}}\right.$$

It is this that is in question. Use the following

<u>Lemma</u> $\frac{1}{2}|\alpha+\beta| + \frac{1}{2}|\alpha-\beta|$

$= \max[|\alpha|, |\beta|]$ for all $\alpha, \beta \in R$.

<u>Proof</u> : Exercise left to you.

We have already given an example to show opt ~~Show~~ solution need not be integral in general.

Finding integer optimal solution to 2-comm. flow problem is NP-Complete (Even, Itai & Shamir) (1976)

However there is a "positive" result as well. due to B. Rothschild and A. Whinston (1966) that shows existence of an optimal solution that is integral if the cut condition holds <u>and</u> the <u>Euler condition</u> (described below) also holds!

$$\sum_{\substack{e \text{ incident} \\ \text{at } v}} C(e) \begin{cases} \equiv 0 \pmod 2 & \text{if } v \neq s_1, s_2, t_1, t_2 \\ \equiv d_1 \pmod 2 & \text{if } v = s_1, t_1 \\ \equiv d_2 \pmod 2 & \text{if } v = s_2, t_2 \end{cases}$$

<u>Proof</u>: If these conditions hold, we may select integral x' and x'' so that

$$x'(e) \equiv C(e) \pmod 2 \ \forall e \in E$$

$$x''(e) \equiv C(e) \pmod 2 \ \forall e \in E$$

To see this, Let $E' = \{e \in E : x'(e) \not\equiv C(e) \pmod 2\}$

$$f(x', v) - f(c, v) \equiv \delta \equiv 0 \pmod 2$$

# of edges incident at $v \in E'$

Hence, if $E' \neq \phi$, E' contains cycles

Increasing flows along the cycle by 1 in some orientation of the cycle, reduces $|E'|$.

Hence the result follows.

$\therefore \frac{1}{2}\left[x'(e)+x''(e)\right]$ and $\frac{1}{2}\left(x'(e)-x''(e)\right)$

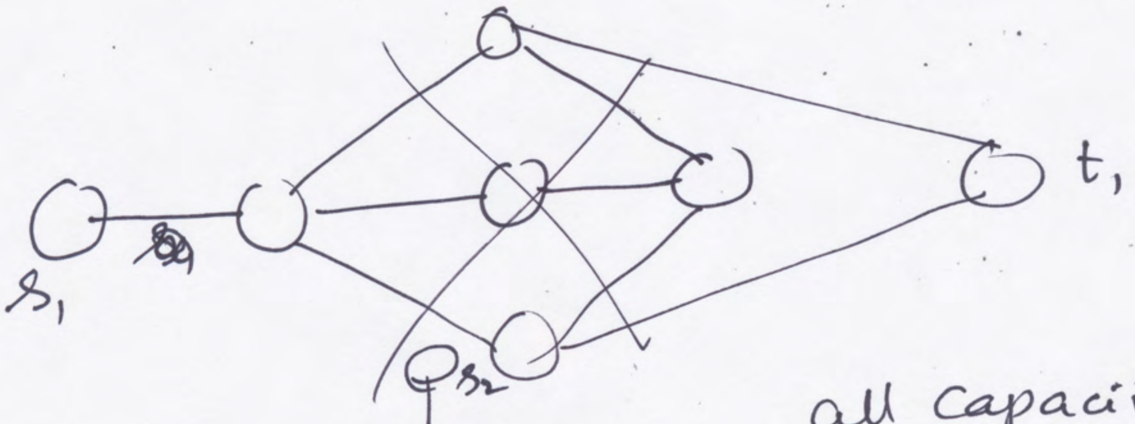
are integers since

$$x' \equiv x'' \equiv c \pmod 2$$

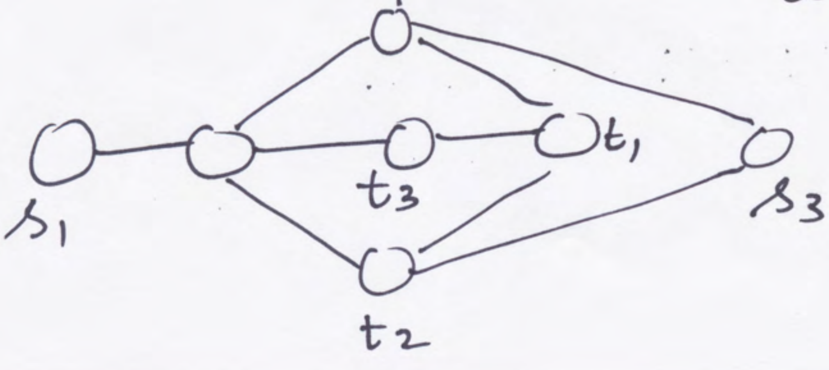
The above proof also gives a strongly poly. time algorithm to produce integer solutions in this case.

When we consider $k=3$, cut conditions are no longer sufficient.

~~See next page~~



all capacities $= 2$.



Suppose ~~$d_1 = 4$, $d_2 = 2$, $d_3 = 1$~~

~~Each is achievable alone; any two~~

Mincut separating all 3 Comm. is 8.

Max-flow is ~~$6\frac{1}{2}$~~ $6\frac{2}{3}$ →

$d_1 = 2$, $d_2 = 2$, $d_3 = 4$

Any two can be achieved. [Work this out!]

But not all three.

Each $s_i - t_i$ path has atleast 3 edges

∴ Each unit of flow of any commodity

Consumes 3 units of capacity.

There are 10 edges → 20 units of total cap.

$$\therefore \text{total flow} \leq \frac{20}{3} = 6\tfrac{2}{3}$$

∴ $d = [2, 2, 4]$ not feasible although it Satisfies cut condition. Moreover, overall flow has $\frac{1}{3}$ fractions.

We Consider General Cases in next lecture