CS6381: Lecture 21 10.17.2020 (1)

Now we look at the Russian Literature e.g. Lomonosov and Karzanov et al.

There are two graphs: $G = [V; E; c]$ – the so called "supply" graph; c: edge capacities ≥ 0

The other called "demand" graph: $H = [T; U; d]$

$T \subseteq V$; $d \geq 0$ demands; edges in U correspond to terminals of commodities;

[In some cases d may not be there as in max-flow problems]

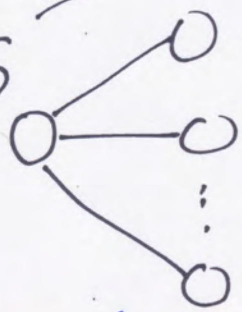
Assumption H has no "isolated" nodes (nodes with no edges of U incident at them)

Most of literature is with both G, H Undirected

but not all

1. Single Commodity: $|U| = 1$.

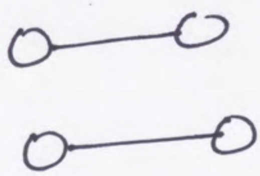
2. $H = S$



"Star" network

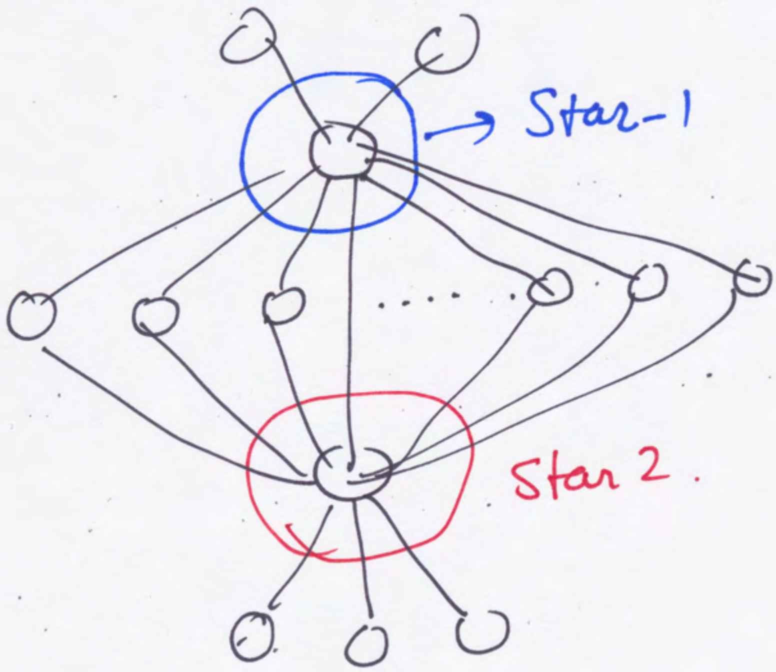
Can easily be converted to 1.

3. H:



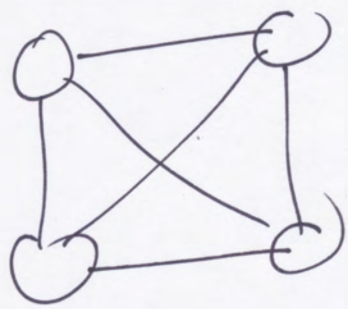
T.C. Hu Problem

4: $H = S^2$: (Shown below) Two-Star



Can be reduced to 3 by the same process used in converting $2 \to 1$.

5. The Simplest case not included in the above is $K_4 = H \subseteq H = K_n$ complete graph



5 was solved by V. L. Kuperschtoch (1971) & Cherkaski, B.V (1976) for max-flow problem: also called "Free-Flow" Problem.

Free flow Problem : (Undirected G version)

Let F_s^* be maximum total flow on G for the star problem $H_s := [T, U_s]$

$$U_s = \{(s,j) : j \in T - s\}$$

"Clearly" $F_s^* = C_s$ where C_s : Capacity of mincut separating s from $T-s$ in G. [The remaining nodes of G may go either side]

f_s^* corresponding F_s^* may be assumed to be integral if C is integral.

Now let us consider Free flow with K_n, $n=4$.

$$\begin{array}{llllllll}
F_{1,2}^* & + F_{1,3}^* & + F_{1,4}^* & & & & \leq & C_1^* \\
F_{1,2}^* & & & + F_{2,3}^* & + F_{2,4}^* & & \leq & C_2^* \\
& F_{1,3}^* & & + F_{2,3}^* & & + F_{3,4}^* & \leq & C_3^* \\
& & F_{1,4}^* & & + F_{2,4}^* & + F_{3,4}^* & \leq & C_4^* \\
\hline
\end{array}$$

$$F_{1,2}^* + F_{1,3}^* + F_{1,4}^* + F_{2,3}^* + F_{2,4}^* + F_{3,4}^* \leq \frac{1}{2} \sum_{i=1}^{4} C_i^*$$

Kupperschtoch showed equality holds.

Which shows this problem is "Cut" dependent. Moreover, we get half integrality if C is integral. and integrality if G is <u>Eulerian</u>

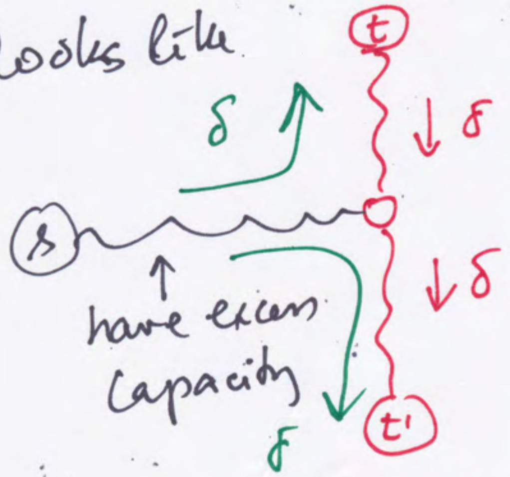
$$\sum_{\substack{e:\text{ incident}\\ \text{at any node}\\ \notin T}} c_e \equiv 0 \pmod 2$$

T- Routine (~~Kuperschtoch~~ Cherkaski):

Suppose in some optimal solution to the free-flow maximization problem, ~~in~~ $F_s < F_s^*$

(F_s: total flow for commodities with s as one end)

$\therefore$ $\exists$ an augmenting path for the s-star problem (if only "other" commodity flows did not block this augmentation) So the scenario looks like



$t \in T$, $t' \in T$, $t \neq s$, $t' \neq s$

total increases by δ

$\therefore$ This can not happen at optimality & hence the result.

Note that this procedure does not change either F_t or $F_{t'}$ but increases F_s.

Thus, we can increase F_i $i \in T$ one by one <u>without</u> reducing others until all are equal to their respective maximum. Hence this algorithm can also be made strongly polynomial.

Now about preserving half integrality/integrality

<u>Def</u>: C (cap. function) is <u>inner Eulerian</u> with respect to a given $G = [V; E]$ and $H = [T; U]$ if C is integral <u>and</u>

$$\sum_{e:\text{ incident at } i} C(e) \equiv 0 \pmod 2 \quad \forall\, i \in V - T = \overline{T}.$$

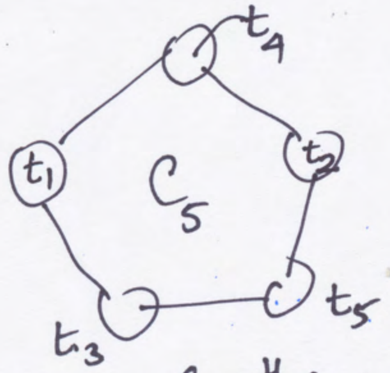
<u>T-routine</u> (Cherkaski)

Throughout this routine, we assume integral flows (starting with zero flows). Consider only flows of the type $F_{s,j}$ $j \neq s$, $j \in T$ for $s \in T$; Keep all other commodity flows integral & frozen, residual capacities satisfy inner Eulerian condition since original cap. dot each path flow reduces by even integer at $j \notin T$.

Hence, there is a double path of unsaturated edges to any node that has such a path at all. This permits us to choose δ in the previous discussion as integral and hence new improved flow is also integral. [Be Careful!]

Next Case

The "Smallest" Case of H not included so far is



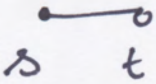
$H = C_5$.

Before, we go further, let us look to what is the underlying "reason" why some of these work. For this we study H more deeply.

See next page

$F_1^* = \cancel{4}\ 2$, $F_2^* = 2$, $F_3^* = 4$

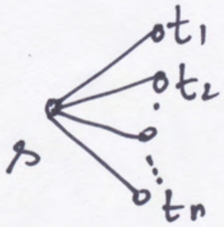
$F_{12}^* = 4$; $F_{1,3}^* = \cancel{4}$



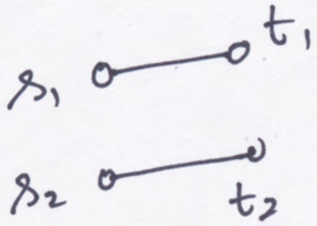
nodes

	s	t
{s}	1	0
{t}	0	1

Anti-chains



	s	t_1	t_2 ...	t_n
{s}	1	0	....	0
$\{t_1, .. t_n\}$	0	1	1 1 1	1



	s_1	t_1	s_2	t_2
$\{s_1, s_2\}$	1	0	1	0
$\{t_1, t_2\}$		1		1
$\{s, t_2\}$	1			1
$\{s_2, t_1\}$	0	1	●	0

Similar for $S^2 = H$

$H = K_n$ $(n=4?)$

	t_1	t_2	t_3	t_4
$\{t_1\}$	1	0	0	0
$\{t_2\}$	0	1	0	0
$\{t_3\}$	0	0	1	0
$\{t_4\}$	0	0	0	1

$H = C_5$

$$\begin{array}{c|ccccc|}
 & t_1 & t_2 & t_3 & t_4 & t_5 \\
\hline
\{t_1, t_2\} & 1 & 1 & 0 & 0 & 0 \\
\{t_2, t_3\} & 0 & 1 & 1 & 0 & 0 \\
\{t_3, t_4\} & 0 & 0 & 1 & 1 & 0 \\
\{t_4, t_5\} & 0 & 0 & 0 & 1 & 1 \\
\{t_5, t_1\} & 1 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

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all of these are <u>cut-dependent</u>. ~~All but~~ 1

First two have integrality property

The next two have $1/2$ integrality property

or integrality for <u>inner Eulerian</u> input.

The last has $1/4$ integrality.

The "Russians" introduced the notion of a <u>fractionality</u> index : ϕ. ϕ is an integer such that if c is integral, solutions are integer multiples of $\frac{1}{\phi}$. (least such integer ϕ).

<u>Russian Conjecture</u> $\exists$ a combinatorial algorithm if and only if fractionality index is finite.

They show, that if Φ is not 1, 2, 4 Then it is Unbounded.

~~The smallest problem not~~

The process of solving involves a procedure called as "Locks and Drains" in Lomonosov's work

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Given $G = [V; E, c]$ → Supply graph

$H = [T; U]$ $T \subseteq V$; U has both ends in T.

Each edge in U is a commodity.

Lomonosov notation: $[X] := K_X$: Complete graph on $X \subseteq V$.

$\|f_u\|$: total flow of Commodity $u \in U$.

We are interested in $\max \sum_{u \in U} \|f_u\|$.

Let $A \subseteq T$: subset of terminal vertices

$F_A^i : \{ f_u : u \in [A] \}$ commodities with both ends in A (<u>internal</u>)

$F_A^e : \{ f_u : u \in [A, \bar{A}] \}$ commodities with one end in A and other in T − A. (external)

{node in A : is origin}

Def A multiflow $\{f, F\}$ <u>locks</u> $A \subseteq V$ if $\exists X \subseteq V$ with $X \cap T = A$ such that

$$\delta_F(A, \bar{A}) = c[X, \bar{X}] \quad — \quad (*)$$

flow ~~from~~ on commodities with one end in A and other not in A, equals cut capacity.

(1) Such a cut must be of min-capacity among all cuts that separate A from $\bar{A}$.

i.e $\min \{ C[Y, \bar{Y}] : Y \subseteq V;\ Y \cap T = A \}$

$:= C\langle A \rangle$. in Lomonosov.

(2) F_A^e is max (<u>multiterminal</u> (single com)) flow in G from A to $T - A$.

(3) $*$ holds iff

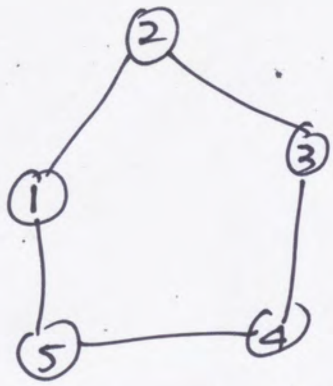
$$\sum_{s \in A} \sum_{t \in \bar{A}} f^{(s,t)}(x, y) = c(x, y) \quad \forall\, x \in X,\ y \in \bar{X}$$

(Each edge across the cut $(X, \bar{X})$ is full)

(4) If $X, \tilde{X}$ satisfy $(*)$ then. So does $X \cap \tilde{X}$.

Mini<u>mal</u> (set theoretically) sets satisfying $(*)$ ~~are~~ is called <u>Locker</u> of A. $\rightarrow \mathcal{L}(A) \rightarrow$ Lomonosov notation

The "idea" is to simultaneously lock all indep. sets in H. We illustrate this using $H = C_5$.

H = 

$|T| = 5$

Look at $\{1,3\}, \{3,5\}, \{5,2\}, \{2,4\}, \{4,1\}$ as potential candidates for A. & look for $\{F_A^{*e}\}^*$

(Each of these can be viewed as multi-origin multi-destination single commodity problems)

$$F^e_{\{1,3\}} = F_{1,2} + F_{2,3} + F_{3,4} + F_{5,1} \leq C^*_{1,3}$$

$$F^e_{\{3,5\}} = F_{4,5} + F_{2,3} + F_{3,4} + F_{5,1} \leq C^*_{3,5}$$

$$F^e_{\{5,2\}} = F_{1,2} + F_{2,3} + F_{4,5} + F_{5,1} \leq C^*_{5,2}$$

$$F^e_{\{2,4\}} = F_{1,2} + F_{2,3} + F_{3,4} + F_{4,5} \leq C^*_{2,4}$$

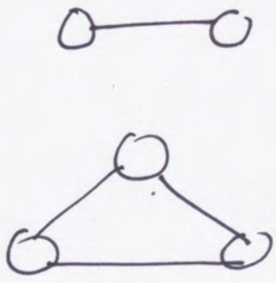
$$F^e_{\{4,1\}} = F_{1,2} + F_{4,5} + F_{3,4} + F_{5,1} \leq C^*_{4,1}$$

$$\text{Total of all commodities} \leq \frac{1}{4}\left[C^*_{13} + C^*_{35} + C^*_{5,2} + C^*_{2,4} + C^*_{4,1}\right]$$

Equality holds at optimality $\therefore$ C_5: cut dependent.

Example that is <u>not</u> cut dependant (smallest)

$H = K_2 + K_3$

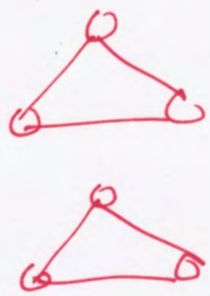


Here Karzanov introduced what he called <u>2-3 metric</u>. & the problem <u>5-terminus flows</u>

(All remaining cases with H a subgraph of K_5 have been already done)

The only other problem that has some chance of being solved well for <u>feasibility</u> is

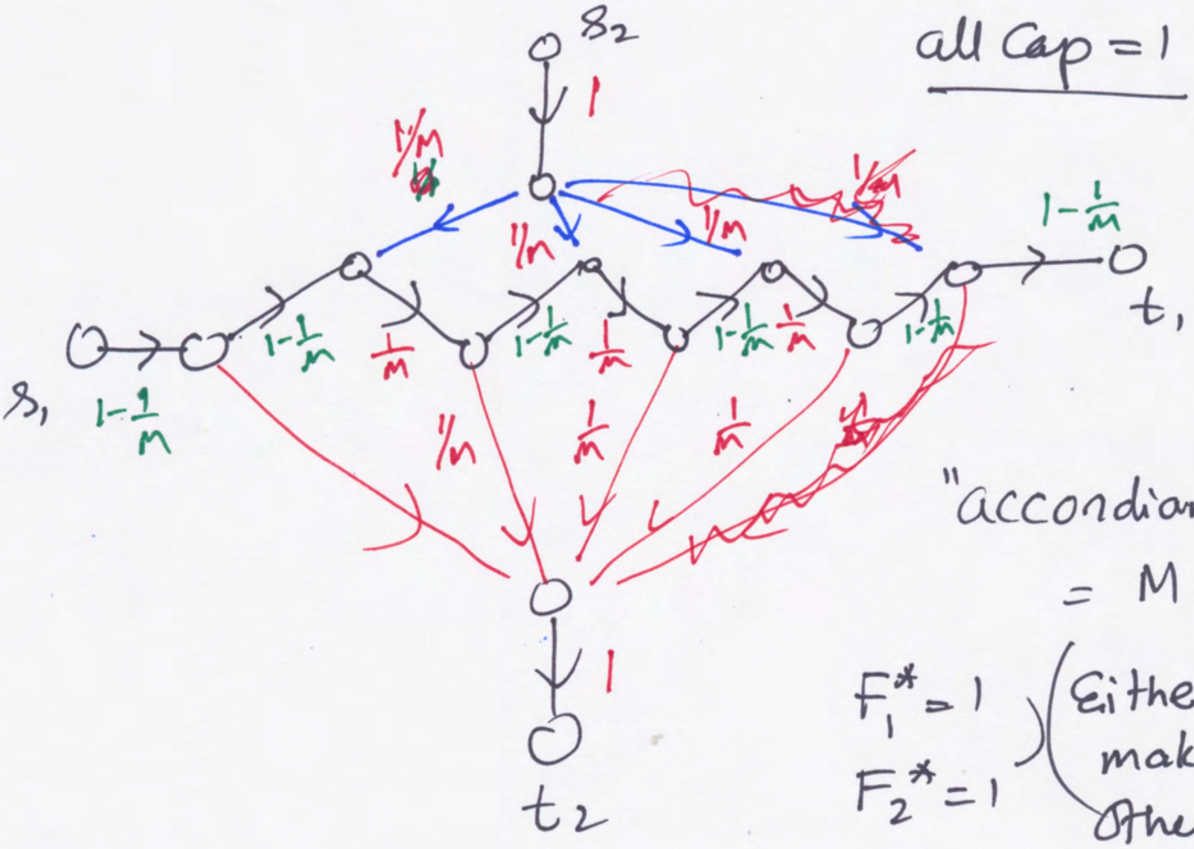
$H = K_3 + K_3$:



These are the only graphs without a matching M of size 3.

→ If this happens <u>Φ is unbounded</u>

W.S. Jewell's example on dir. G. & unbounded fractionality for 2 Commodities



"accordian" Size = M.

$F_1^* = 1$
$F_2^* = 1$
(Either one makes the other = 0 in combined flow.)

$F_1 = 1 - \frac{1}{M}$
$F_2 = 1$
} optimal multi. comm flow