Now we look at the Russian literature e.g. Lomonosov and Koganov et al.

There are two graphs: $G = (V; E; c)$ - the so-called "supply" graph; $c$: edge capacities $\geq 0$

The other called "demand" graph: $H = (T; U; d)$

$T \subseteq V$; $d \geq 0$ demands; edge in $U$ correspond to terminals of commodities;

[In some cases, $d$ may not be there as in max-flow problems]

Assumption $H$ has no "isolated" nodes (nodes with no edges of $U$ incident at them)


2. $H = S$

   Star network

   Can easily be converted to 1.

3. $H$:

   T.C. Hu Problem
4. $H = S^2$: (Shown below) **Two-Star**

Can be reduced to 3 by the same process used in converting $2 \rightarrow 1$.

5. The simplest case not included in the above is $K_4 \leq H \leq K_n$ **Complete graph**

5 was solved by V. L. Kuperschtoch (1971) and Chenkaski, B. v (1976) for max-flow problem! Also called "Free-Flow" Problem.
Free flow Problem: (Undirected G version)

Let $F^*_S$ be maximum total flow on $G_s$ for the
Star problem $H_s := [T, U_s]

\quad \quad \quad \quad \quad U_s = \{ (s, i) : i \in T-S \}$

"Clearly" $F^*_S = c_s$ where $c_s$ : Capacity $C$

"min-cut separating $s$ from $T-S$ in $G_s$. [The

Remaining node $s'$ of $G_s$ may go either side.]

$s'$ Corresponding $F^*_S$ may be assumed to be

Integral if $C$ is integral.

Now let us consider Free flow with $K_n, n=4$.

\[
\begin{align*}
F^*_{1,2} + F^*_{1,3} + F^*_{1,4} & \leq C^*_1 \\
F^*_{1,2} + F^*_{2,3} + F^*_{2,4} & \leq C^*_2 \\
F^*_{1,3} + F^*_{2,3} + F^*_{3,4} & \leq C^*_3 \\
F^*_{1,4} + F^*_{2,4} + F^*_{3,4} & \leq C^*_4 \\
\end{align*}
\]

\[
\begin{align*}
F^*_{1,2} + F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4} + F^*_{3,4} & \leq \frac{1}{2} \sum_{i=1}^{4} C^*_i \\
\end{align*}
\]

It is shown equality holds.
Which shows this problem is "Cut" dependent. (4)
Moreover, we get half integrality if \( C \) is integral
and integrality if \( G \) is Eulerian
\[
\sum C_e \equiv 0 \pmod{2}
\]
\( e: \) incident at any node \( e \in T \)

**T- Routine (Kuperschloch)**:

Suppose in some optimal solution to the free-flow maximization problem, \( F < F^* \)
(\( F_0 \): flow for commodities with \( s \) as one end)

\( \exists \) an augmenting path for the \( s \)-star problem
(if only "other" commodity flows did not block this augmentation) so the scenario looks like

\[
\begin{align*}
& \delta \downarrow \quad \delta \downarrow \\
& \delta \uparrow \quad \delta \uparrow
\end{align*}
\]
Note that this procedure does not change either $F_t$ or $F_s$, but increases $F_s$.

Thus, we can increase $F_i$ in $T$ one by one without reducing others until all are equal to their respective maximum. Hence this algorithm can also be made strongly polynomial.

Now about preserving half integrality/integrality

**Def.** $C$ (cap. function) is **inner Eulerian** with respect to a given $G=\langle V; E \rangle$ and $H=\langle T; U \rangle$ if $C$ is integral and

$$\sum C(e) \equiv 0 \pmod{2} \quad \forall \; e \in V-T=T.$$

$e$: incident at $i$

**T-routine (Cherkasski)**

Throughout this routine, we assume integral flows (starting with zero flows). Consider only flows of the type $E_{3,j}$, $j \neq s$ in $T$ for each $s \in T$; keep all other commodity flows integral & frozen, residual capacities satisfy inner Eulerian condition since original cap. that each path flow reduces by even integer at $j \in T$.
Hence, there is a double path of unsaturated edge \( v \) to any node that has such a path at all. This permits us to choose \( \delta \) in the previous discussion as integral and hence new improved flows also integral. [Be careful!]

**Next Case**
The "Smallest" case of \( H \) not included so far is

\[
H = C_5
\]

Before we go further, let us look to what is the underlying "reason" why some of these work. For this we study \( H \) more deeply.

See next page
\[ F_1^* = 4, \quad F_2^* = 2, \quad F_3^* = 4 \]

\[ F_1^* = 4, \quad F_2^* = 4 \]

\[ s \quad t \]

\[ \begin{array}{c|c}
  s & t \\
  \hline
  s & 1 \\
  t & 0 \\
  s & 0 \\
  t & 1 \\
\end{array} \]

Anti-chrom

\[ \begin{array}{c|c|c|c|c|c|c|c}
  & t_1 & t_2 & \ldots & t_n \\
  \hline
  s & 1 & 0 & \ldots & 0 \\
  t_1 & 0 & 1 & 1 & 1 & 1 \\
  t_2 & 0 & 1 & 1 & 1 & 1 \\
  \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  t_n & 0 & 1 & 1 & 1 & 1 \\
\end{array} \]

Similar for \( S^2 = H \)

\[ H = K_n \quad (n=4?) \]

\[ \begin{array}{c|c|c|c|c|c|c|c}
  & t_1 & t_2 & t_3 & t_4 \\
  \hline
  t_1 & 1 & 0 & 0 & 0 \\
  t_2 & 0 & 1 & 0 & 0 \\
  t_3 & 0 & 0 & 1 & 0 \\
  t_4 & 0 & 0 & 0 & 1 \\
\end{array} \]
All of these are cut-dependent. At least:

First two have integrality property.
The next two have $\frac{1}{2}$ integrality property or integrality for inner Eulerian input.

The last has $\frac{1}{4}$ integrality.

The "Russians" introduced the notion of a fractionality index $\phi$. $\phi$ is an integer such that $K \cdot c$ is integral, solutions are integral multiples of $\frac{1}{\phi}$. (Least such integer $\phi$).

**Russian Conjecture** $\exists$ a combinatorial algorithm if and only if fractionality index is finite.
They show that if $\Phi$ is not 1, 2, 4 then it is unbounded.

The smallest problem not

The process of solving involves a procedure called "Links and Drains" in Lomonosov's work.

---

**Given** $G = [V; E, \mathcal{E}] \rightarrow$ Supply graph

$T \subseteq V; \ T \in T; \ U \ has \ both \ ends \ in \ T.$

Each edge in $U$ is a commodity.

Lomonosov notation: $[x] = K_x \ : \ complete \ graph \ on \ x \in V.$

$\|f_u\| = \text{total flow of commodity } u \in U.$

We are interested in $\max \sum_{u \in U} \|f_u\|.$

Let $A \subseteq T \ : \ Subset \ of \ terminal \ vertices$ commodities with both ends in $A$ (internal)

$F^i_A = \{f_u : u \in [A]\}$

$F^e_A = \{f_u : u \in [A, \bar{A}]\}$ commodities with one end in $A$ and other in $T-A.$ (external)

$\sqrt{\{node \ in \ A: \ is \ origin\}}$
A multigflow \( \{f, F\} \) locks \( A \subseteq V \) if \( \exists X \subseteq V \) with \( X \cap T = A \) such that
\[
\delta_F(A, \overline{A}) = \epsilon \left[ x, \overline{x} \right] 
\]
flow from commodities with one end in \( A \) and other not in \( A \), equals cut capacity.

1) Such a cut must be of min-capacity among all cuts that separate \( A \) from \( \overline{A} \).
   i.e. \( \min \{ C [Y, \overline{Y}] ; Y \subseteq V ; Y \cap T = A \} \)
   \( \leq C \langle A \rangle \) in Lomonosov.

2) \( F^e_A \) is \( \max \) (multiterminal) flow in \( G \) from \( A \) to \( T - A \).

3) it holds iff
\[
\sum_{x \in A} \sum_{y \in \overline{A}} f_{x,y}^{(s,t)} = c(x,y) \quad \forall x \in X \quad y \in \overline{x}
\]
   (Each edge across the cut \( (X, \overline{X}) \) is full)

4) \( \forall X, \overline{X} \) satisfy \((a)\) then \( \exists \text{ other } X \cap T \neq \emptyset \).

Minimal (set theoretically) sets satisfying \((a)\) are called **lockers of \( A \)**. \( \mathcal{L}(A) \) Lomonosov notation.
The "idea" is to simultaneously lock all independent sets in $H$. We illustrate this using $H = C_5$.

$$H = \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{array}$$

$$|T| = 5$$

Look at $\{1, 3\}, \{3, 5\}, \{5, 2\}, \{2, 4\}, \{4, 1\}$ as potential candidates for $A$. We look for $F_A^{\text{opt}}$ (Each of these can be viewed as multi-origin multi-destination single commodity problem)

$$F_e^{\{1, 3\}} = F_{1, 2} + F_{2, 3} + F_{3, 4} + F_{5, 1} \leq C_{1, 3}^\text{opt}$$

$$F_e^{\{3, 5\}} = F_{4, 5} + F_{2, 3} + F_{3, 4} + F_{5, 1} \leq C_{3, 5}^\text{opt}$$

$$F_e^{\{5, 2\}} = F_{1, 2} + F_{2, 3} + F_{4, 5} + F_{5, 1} \leq C_{5, 2}^\text{opt}$$

$$F_e^{\{2, 4\}} = F_{1/2} + F_{2, 3} + F_{3, 4} + F_{4, 5} \leq C_{2, 4}^\text{opt}$$

$$F_e^{\{4, 1\}} = F_{1/2} + F_{4, 5} + F_{3, 4} + F_{5, 1} \leq C_{4, 1}^\text{opt}$$

The total of all commodities is:

$$\frac{1}{4} \left[ C_{1, 3}^\text{opt} + C_{3, 5}^\text{opt} + C_{5, 2}^\text{opt} + C_{2, 4}^\text{opt} \right]$$

Equality holds at optimality on $C_5$: cut dependent.
Example that is not cut dependent (smallest)

\[ H = K_2 + K_3 \]

Here Karzanov introduced what he called 2-3 metric & the problem 5-terminus flows

(All remaining cases with \( H \) a subgraph of \( K_5 \) have been already done)

The only other problem that has some chance of being solved well for feasibility is

\[ H = K_3 + K_3 \]

These are the only graphs without a matching \( M \) of size 3.

\( \rightarrow \) If this happens \( Q \) is unbounded
W.S. Jewell's example on dir. G.2 unbounded fractionality for 2 Commodities

all Cap = 1

"accordion" size = M

$F_1^* = 1$ (Either one make the other = 0 in combined flow)

$F_2^* = 1$

$F_1 = 1 - \frac{1}{M}$

$F_2 = 1$

$\{ \text{optimal multi. comm. flow} \}$