Blocking Systems & Polyhedra

(Also Antiblocking polyhedra)

<u>Definition</u>: Let E be a (finite?) set. Let $\mathcal{P}$ be a family of subsets of E such that

$\cancel{P_1 \in \mathcal{P}}, \cancel{P_2 \in \mathcal{P}} \;\cancel{\Rightarrow}$

$P_1 \in \mathcal{P},\ P_2 \subsetneq P_1 \Rightarrow P_2 \notin \mathcal{P}$ (non nested family)

Then $\mathcal{P}$ is called a clutter on E.

<u>Definition</u>: Let $\mathcal{P}$ and $\mathcal{K}$ be clutters on E satisfying the property:

$\forall E^0 \subseteq E$ <u>exactly</u> <u>one</u> of the following holds:

(i) $\exists P \in \mathcal{P}$ such that $P \subseteq E^0$

(ii) $\exists K \in \mathcal{K}$ such that $K \subseteq E - E^0 = \bar{E}^0$.

Then we call the triple $(E, \mathcal{P}, \mathcal{K})$ a blocking system. Note that this automatically also implies $(E, \mathcal{K}, \mathcal{P})$ is a blocking system.

Each member of $\mathcal{K}$ is said to <u>block</u> each member of $\mathcal{P}$ and conversely.

Lemma 1: If $[E, \mathcal{P}, \mathcal{K}]$ is a blocking system, then

$$P \cap K \neq \phi \quad \forall P \in \mathcal{P}, \forall K \in \mathcal{K}. \quad (*)$$

Conversely, if $\mathcal{P}, \mathcal{K}$ are clutters on E, satisfying $(*)$, then both (i) & (ii) in the definition of a blocking system can not be true.

Pf: Let $[E, \mathcal{P}, \mathcal{K}]$ be a blocking system. If $\exists$ $P \in \mathcal{P}$, $K \in \mathcal{K}$ such that $P \cap K = \phi$,

Then letting $E^0 = P$ satisfies,

$\exists P \in \mathcal{P}$, $P \subseteq E^0$ & $\exists K \subseteq \overline{E}^0$, $K \in \mathcal{K}$.

Violating exactly one of {(i) + (ii)} is true.

Converse part of the lemma is easy to prove.

Theorem 2: Let $\mathcal{P}$ be a clutter on E. $\exists$ a unique clutter $\mathcal{K}$ on E such that $(E, \mathcal{P}, \mathcal{K})$ is a blocking system.

Pf: First we construct such a $\mathcal{K}$ and then we show it is unique by proof by contradiction.

Construction of a $\mathcal{K}$:

Let $\mathcal{F} = \{ S \subseteq E : S \cap P \neq \phi \ \forall P \in \mathcal{P} \}$

Note $\mathcal{F}$ is not empty; $(S = E)$!

Let $\mathcal{K} = \{ K \in \mathcal{F} : K' \subsetneq K \Rightarrow K' \notin \mathcal{F} \}$

minimal (set theoretically) members in $\mathcal{F}$ are in $\mathcal{K}$.

Hence $K \cap P \neq \phi \ \forall K \in \mathcal{K} . \forall P \in \mathcal{P}$.

Hence both (i) and (ii) in the definition of blocking system can not be true simultaneously.

Clearly, $\mathcal{K}$ is a clutter on E. Suppose for some $E^0 \subseteq E$, (i) is not true i.e $\nexists P \in \mathcal{P}$, $P \subseteq E^0$. Then $\forall P \in \mathcal{P}$, $(E - E^0) \cap P \neq \phi$. and therefore $E - E^0 \in \mathcal{F}$. Hence some set $K \in \mathcal{K}$ is such that $K \subseteq E - E^0$.

Hence $(E, \mathcal{P}, \mathcal{K})$ is a blocking system.

Next we take up uniqueness of $\mathcal{K}$.

Suppose $(E, \mathcal{P}, \mathcal{K})$ and $(E, \mathcal{P}, \mathcal{K}')$ are both blocking systems & $\mathcal{K} \neq \mathcal{K}'$.

Let $K \in \mathcal{K} - \mathcal{K}'$. Consider Partition $(E-K, K)$ of E.

$\nexists$ (iii) is satisfied. $\therefore \nexists P \in \mathcal{P}, P \subseteq E-K$ (using $(E, \mathcal{P}, \mathcal{K})$ is a blocking system)

Now consider the blocking system $(E, \mathcal{P}, \mathcal{K}')$

Since $\nexists P \in \mathcal{P}, P \subseteq E-K$, for the partition $(E-K, K)$, $\exists K' \in \mathcal{K}', \underbrace{K' \subsetneq K}_{\Rightarrow K' \subset K.}$ (Since $K \in \mathcal{K} - \mathcal{K}'$).

Now consider the partition $(E-K', K')$

Since (iii) is satisfied for $(E, \mathcal{P}, \mathcal{K}')$, for this partition, $\nexists P \in \mathcal{P}, P \subseteq E-K'$.

Now use system $(E, \mathcal{P}, \mathcal{K})$ on $(E-K', K')$

Since $\nexists P \in \mathcal{P}, P \subseteq E-K'$, $\exists K'' \in \mathcal{K}, K'' \subseteq K'$

Note $K'' \subseteq K' \subset K$ $\quad \underbrace{K'' \in \mathcal{K}, K \in \mathcal{K}}$

$\mathcal{K}$ is not a clutter on E. Hence the uniqueness ~~of~~ result.

So clutters on E ~~or~~ are paired to form various blocking systems.

Now as to why we are interested in blocking systems.

<u>Theorem 2</u>: Let $(E, \mathcal{P}, \mathcal{K})$ be a blocking system. Let $f : E \to R$ be a function on E. Then

<u>Min Max equality</u>

Bottleneck Problems ←

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = \min_{K \in \mathcal{K}} \max_{e \in K} f(e) \quad (**)$$

Conversely, if $(**)$ holds (even for 0/1 valued f) for a pair of Clutters $\mathcal{P}$, & $\mathcal{K}$ on E, $(E, \mathcal{P}, \mathcal{K})$ is a blocking system.

<u>Proof</u>: If $(E, \mathcal{P}, \mathcal{K})$ is a blocking system, since $P \cap K \neq \phi \;\; \forall P \in \mathcal{P}, \forall K \in \mathcal{K}$, it is easy to verify that

$$\min_{e \in P} f(e) \leq f(\hat{e}) \leq \max_{e \in K} f(e) \qquad \forall \, P \in \mathcal{P}, \; K \in \mathcal{K}$$

(when $\hat{e} \in P \cap K$. Hence

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) \leq \min_{K \in \mathcal{K}} \max_{e \in K} f(e). \qquad (\mathrm{I})$$

Since P ~~and~~, $P \in \mathcal{P}$ have finitely many elements, let

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = \min_{e \in P^*} f(e) = f(e^*)$$

Define $E^0 = \{e \in E : f(e) > f(e^*)\}$

$$\hat{E}^0 = \{e \in E : f(e) \geqq f(e^*)\}$$

By the above, $\exists\, P \in \mathcal{P}$, such that $P \subseteq \hat{E}^0$ (say P^*) and hence $\nexists\, K \subseteq E - \hat{E}^0$

But $\nexists\, P \in \mathcal{P}$, such that $P \subseteq E^0$ hence $\exists\, K^* \subseteq E - E^0$. The equation in theorem holds for P^*, K^*. (in I).

<u>To show Converse</u>: Let $E^0 \subseteq E$. Define f as follows

$$f(e) = \begin{cases} 1 & e \in E^0 \\ 0 & e \notin E^0 \end{cases}$$

If $\exists P \subseteq E^0$, then

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = 1 = \min_{K \in \mathcal{K}} \max_{e \in K} f(e)$$

implying $K \cap E^0 \neq 0 \ \forall K \in \mathcal{K}$.

Hence $\nexists\, k \in \mathcal{K}$ such that $k \subseteq E - E^0$.

If $\nexists\, P \in \mathcal{P}, P \subseteq E^0$, then

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = 0 = \min_{k \in \mathcal{K}} \max_{e \in k} f(e)$$

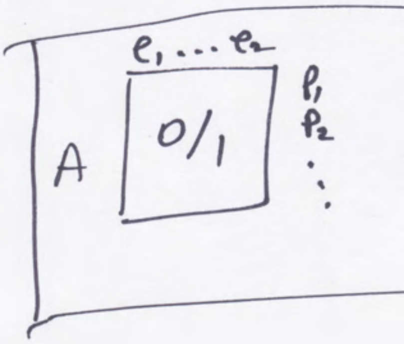
This in turn implies $\exists\, k \in \mathcal{K}, k \subseteq E - E^0$.

Hence the converse is also true and hence the theorem follows.

Max-flow-(Min-cut) Equality

Let A be the incidence matrix of Elements of E (columns of A) and members of $\mathcal{P}$ (rows of A)

$$a(P,e) = \begin{cases} 1 & \text{if } e \in P \\ 0 & \text{else.} \end{cases}$$



Consider the Linear Program

$$\max \sum_{P \in \mathcal{P}} y(P)$$

$$y \geq 0$$

$$\sum_{P \in \mathcal{P}} a(P,e)\, y(P) \leq w(e) \quad e \in E$$

$W(e)$: Capacity of element e

Special Case E set of edges in a ~~directed~~ graph G

$\mathcal{P}$: set of all s-t paths in G

$\mathcal{K}$: set of "cuts" separating s and t.

(Think minimal sets of edges whose removal leaves no s-t path)

LP: max-flow problem in path-edge formulation

related problem $\min\limits_{K \in \mathcal{K}} \sum\limits_{e \in K} w(e) \Leftrightarrow$ min-cut

A blocking system is said to satisfy max-flow-min-cut ~~prop~~ equality if

$$\max \sum y(p) : y \geqq 0, \sum_{p \in \mathcal{P}} a(p,e)\, y(p) \leqq w(e) \;\forall e \in E$$

path packing problem

$$\| $$

$$\min_{K \in \mathcal{K}} \sum_{e \in K} w(e)$$

If has integral solutions, then MFMC equality is satisfied Strongly

If we reverse the roles of $\mathcal{P}$, and $\mathcal{K}$, we say we get min path problem & and min-path equality

More precisely:

$$\max \sum_{K \in \mathcal{K}} y(K)$$

→ Cut packing problem

$$y \geq 0$$

$$\sum_{K \in \mathcal{K}} a(K,e)\, y(K) \leq l(e) \quad \forall e \in P$$

min path equality

"dual"

$$\min_{P \in \mathcal{P}} \sum_{e \in P} l(e) \rightarrow \underline{\underline{SP}}$$

Lehman's Length-Width Inequality

Let $(E, \mathcal{P}, \mathcal{K})$ be a blocking system. Let $l : E \to R_+$ and $w : E \to R_+$ be two functions

Let $\Lambda = \min_{P \in \mathcal{P}} \sum_{e \in P} l(e)$

& $\Omega = \min_{K \in \mathcal{K}} \sum_{e \in K} w(e)$

We say that <u>length-width</u> <u>inequality</u> holds for a blocking system $(E, \mathcal{P}, \mathcal{K})$ <u>if</u>

$$\Lambda\, \Omega \leq \sum_{e \in E} l(e)\, w(e) \quad \forall\, l, w \in \mathbb{R}_+^E.$$

Clearly, if length-width inequality holds for a blocking system, $(E, \mathcal{P}, \mathcal{K})$ it also holds for the blocking system $(E, \mathcal{K}, \mathcal{P})$.

<u>Main result</u>

<u>Theorem 3</u>: Let $(E, \mathcal{P}, \mathcal{K})$ be a blocking system. Then either all of the following hold ~~or~~ or none of them hold.

(i) max-flow-min-cut for $(E, \mathcal{P}, \mathcal{K})$

(ii) max-flow-min-cut for $(E, \mathcal{K}, \mathcal{P})$

(iii) Length-width inequality for $(E, \mathcal{P}, \mathcal{K})$ (and of course $(E, \mathcal{K}, \mathcal{P})$)

<u>Pf</u> Suffices to show (i) $\Leftrightarrow$ (iii).

We use LP duality:

(i) $\Rightarrow$ (iii)

Let $L(P) = \sum_{e \in P} l(e)$; let y^* be optimal to

the linear program

$$\max \sum_{P \in \mathcal{P}} y(P) \underset{(i)}{\Longrightarrow} \min_{K \in \mathcal{K}} \sum_{e \in K} w(e) = \Omega$$

$$y \geq 0$$

$$\sum_{P \in \mathcal{P}} a(P,e)\, y(P) \leq w(e) \quad \forall e \in E \qquad (**)$$

$$\Lambda \Omega = \Lambda \sum_{P \in \mathcal{P}} y^*(P) \qquad \text{Since } \Lambda = \min_{P \in \mathcal{P}} L(P)$$

$$\leq \sum_{P \in \mathcal{P}} L(P)\, y^*(P)$$

$$= \sum_{P \in \mathcal{P}} y^*(P) \sum_{e \in P} l(e)$$

def of $a(P,e)$

$$= \sum_{P \in \mathcal{P}} y^*(P) \sum_{e \in E} a(P,e)\, l(e)$$

$$= \sum_{e \in E} l(e) \sum_{P \in \mathcal{P}} a(P,e)\, y^*(P) \leq \sum_{e \in E} l(e)\, w(e)$$

by $(**)$

(iii) $\Rightarrow$ ii)

(i) $$\text{Max} \sum_{P \in \mathcal{P}} y(P)$$

P

s.t.: $y \geq 0$

$$l(e) \geq 0 : \quad \sum_{P \in \mathcal{P}} a(P,e)\, y(P) \leq w(e) \quad \forall e \in E$$

$$\text{Min} \sum_{e \in E} w(e)\, l(e)$$

D

s.t $l \geq 0$

$$\sum_{e \in E} \cancel{l(e)}\, a(P,e)\, l(e) \geq 1 \quad \forall P \in \mathcal{P}$$

$$= L(P) \geq 1 \quad \forall P \in \mathcal{P}$$

Let the opt. Sol to D be $\{l^*(e)\}$, opt to P be $\{y^*(P)\}$

<u>Claim</u>: $$\Omega = \min_{K \in \mathcal{K}} \sum_{e \in K} w(e) \geq \sum_{P \in \mathcal{P}} y^*(P)$$

For this, let $l_0(e) = \begin{cases} 1 & \text{for } e \in K_0 \\ 0 & \text{else} \end{cases}$

for some $K_0 \in \mathcal{K}$.

Since $P \cap K_0 \neq \phi \ \forall P \in \mathcal{P}$, it is clear that $\{l_0(e)\}$ is a feasible solution to the Dual

Hence by weak duality, it follows that

$$\sum_{e \in K_0} w(e) \geq \sum_{P \in \mathcal{P}} y^*(P)$$

K_0 was arbitrary ;

Hence equality holds & hence (i) holds

$$\Omega = \min_{K \in \mathcal{K}} \sum_{e \in K_0} w(e) \geq \sum_{P \in \mathcal{P}} y^*(P) = \sum_{e \in E} l^*(e)\, w(e)$$

where $\{l^*(e)\}$ is opt to dual.

If strict inequality holds

Now let $\Delta^* = \min_{P \in \mathcal{P}} \sum_{e \in P} l^*(e)$

Since $\{l^*(e)\}$ satisfies

$$\sum_{e \in E} a(P,e)\, l^*(e) \geq 1 \quad \forall P \in \mathcal{P}$$

$$\Downarrow$$

$$L^*(P) \geq 1 \quad \forall P \in \mathcal{P}$$

Hence $\Delta^* = \min_{P \in \mathcal{P}} L^*(P) \geq 1$

$\therefore$ Yields a contradiction to LW Inequality.