CS6381: lecture 24 10.28.2020 (1)

Where does this lead us? . It leads us to what are called Set-Packing Problems and Set-Covering problems.

Set packing problems

Let E be a finite set; Let $\mathcal{P}$ be a family of subsets of E. & Let ${}_{m}\overset{n}{A}$ be an incidence matrix (0-1 entries) where ~~columns~~ rows represent $P \in \mathcal{P}$, ~~rows~~ columns represent $e \in E$. Let $w(e), e \in E$ be non-negative (possibly integer) weights (Capacity of $e \in E$) Consider, the <u>integer</u> Linear Program.

$$\text{Max} \sum_{P \in \mathcal{P}} y(P) = \sum_{i=1}^{m} y_i$$

$$\text{s.t. } A^t y \leq w$$

$$y \geq 0, \text{ integer.}$$

This is called a set packing problem.

In the same Context, the integer LP

$$\text{Min } \sum y_i$$
$$A^t y \geqq w$$
$$y \geqq 0, \text{ integer}$$

is called a Set-covering problem.

Examples:

Coloring problems

Edge-coloring : Given an undirected graph $G = [V; E]$, a valid edge-coloring gives colors to edges so that two edges incident at a vertex receive different colors

We want min # of colors required

Note Edges that receive the same color form a matching (if coloring is valid)

$$A : \begin{array}{cc} & \text{Matching} \\ \text{edge} & \boxed{} \end{array} ; \quad \left.\begin{array}{l} y \geqq 0, \text{ integer} \\ Ay \geqq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ \text{Min } \sum_j y_j \end{array}\right\}$$

Set - Covering

For node coloring, we must use <u>independent Sets instead of Matching</u> → again a Set Covering problem

<u>Max flow</u>

$$A = \begin{array}{c|c} & \text{s-t Paths} \\ \hline \text{edges} & \boxed{} \end{array} \; ; \qquad \begin{aligned} & y \geq 0 \\ & Ay \leq c \\ & \text{Max} \sum_j y_j \end{aligned}$$

Set ~~path~~ packing;

<u>Path-Packing Problem</u>

<u>Shortest Path</u>: Cut-Packing Problem

This leads via Linear Programming duality to

<u>Blocking and Anti-Blocking Polyhedra</u>

which we take up next.

Blocking Polyhedra:

Set Packing

$$\text{Max } \sum y_j = e^t y$$

$$x \geq 0: \quad A^t y \leq w$$

$$y \geq 0, (\text{int})$$

(P) viewed as LP.

$$\text{Min } w^t x$$

$$Ax \geq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = e \qquad (D)$$

$$x \geq 0$$

Set of feasible solutions to dual is an unbounded convex polyhedron and is denoted by $\mathcal{B}$. $\mathcal{B}$ can be expressed as a sum of convex-hull of its extreme points and non-neg. orthant R_+^n.

Def: i^{th} row is <u>inessential</u> if $\exists\ \lambda$ such that

$$\lambda \geq 0, \ ; \ A_{i\cdot} \geq \sum_{j \neq i} \lambda_j A_{j\cdot} \quad \sum_{j \neq i} \lambda_j = 1$$

(i^{th} constraint is redundant in (D))

Def: A is <u>proper</u> if $\nexists$ inessential rows.

(equivalent to non-nestedness of sets)

<u>Def</u> The <u>blocker</u> $\hat{\mathcal{B}}$ of $\mathcal{B}$ is defined by:

$$\mathcal{B} = \{ y : y \geqq 0, y^t x \geqq 1 \; \forall \; x \in \mathcal{B} \}$$

(Equivalent to $P \cap K \neq \phi \; \forall \; P \in \mathcal{P}, \forall K \in \mathcal{K}$)

Let extreme points of $\mathcal{B}$ be $b^1, b^2, \ldots b^r$

Let B be a $r \times n$ matrix whose rows are $b^1, b^2, \ldots b^r$. Let $\mathcal{A} = \{ y : y \geqq 0, By \geqq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \}$

Then (i) $\mathcal{A} = \hat{\mathcal{B}}$.

(ii) B is proper assuming A is proper.

ext pts of $\mathcal{A}$ are rows of A.

(iii) $\hat{\mathcal{A}} = \mathcal{B}$.

(i.e. Blocker of blocker is the original system)

$$\hat{\mathcal{A}} = \{ x : x \geqq 0 , x^t y \geqq 1 \; \forall y \in \mathcal{A} \}$$

We can extend all of the results in Blocking (6)
Systems to Blocking Polyhedra.