Where does this lead us? It leads us to what are called Set-Packing Problem and Set-Covering problem.

Set packing problems

Let $E$ be a finite set; let $\mathcal{P}$ be a family of subsets of $E$. Let $\mathbf{A}$ be an incidence matrix (0-1 entries) where columns represent PEP, rows represent $e \in E$. Let $w(e), e \in E$ be non-negative (possibly integer) weights (Capacity of $e \in E$). Consider the integer linear program:

$$\text{Max } \sum_{\mathbf{p} \in \mathcal{P}} y(\mathbf{p}) = \sum_{i=1}^{m} y_i$$

s.t. $\mathbf{A}^t \mathbf{y} \leq \mathbf{w}$

$y \geq 0$, integer.

This is called a set packing problem.
In the same context, the integer LP

\[
\begin{align*}
\min & \sum y_i \\
A^t y & \geq w \\
y & \geq 0, \text{ integer}
\end{align*}
\]

is called a set-covering problem.

**Example:**

**Coloring problems**

**Edge-coloring**: Given an undirected graph \( G = [V, E] \), a valid edge-coloring gives colors to edges so that two edges incident at a vertex receive different colors.

We want \( \min \# \text{ of colors required} \)

**Note**: Edges that receive the same color form a matching (if coloring is valid).

\[
\begin{align*}
A & = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad y & \geq 0, \text{ integer} \\
A y & \geq (1, 1) \\
\min & \sum y_j
\end{align*}
\]
For node coloring, we must use independent sets instead of matching → again a set covering problem.

Max flow

\[ A = \begin{bmatrix} \end{bmatrix}; \quad A y \leq c, \quad \text{Max } \sum y_j \]

Set path packing;

Path-Packing Problem

Shortest Path: Cut-Packing Problem

This leads via linear programming duality to Blocking and Anti-Blocking Polyhedra

Which we take up next.
Blocking Polyhedra.

Set Packing

\[
\text{Max } \sum y_j = e^T y \\
x_j \geq 0; \quad A^T y \leq w \\
y \geq 0, (\text{int})
\]

\[\text{Min } w^T x \]

\[A x \geq (\frac{1}{w}) = e \]

\[x \geq 0\]

Set of feasible solutions to dual is an unbounded convex polyhedron and is denoted by \(\mathcal{B}\). \(\mathcal{B}\) can be expressed as a sum of convex-hull of its extreme points and non-negative orthant \(\mathbb{R}^n_+\).

Def: \(i\)th row is inessential if \(\exists \lambda\) such that

\[
\lambda \geq 0, \quad A_i \cdot \geq \sum_{j \neq i} \lambda_j A_j, \quad \sum_{j \neq i} \lambda_j = 1
\]

(i \text{th constraint is redundant if } (D))
Def: A is proper if \( \mathcal{X} \) is inessential rows. (equivalent to non-nestedness of sets)

Def: The blocker \( \hat{\mathcal{B}} \) of \( \mathcal{B} \) is defined by:
\[
\hat{\mathcal{B}} = \{ y : y \geq 0, y^T x \geq 1 \forall x \in \mathcal{B} \}
\]
(Equivalent to \( \mathcal{P} \neq \emptyset \forall \mathcal{P} \in \mathcal{Q}, \forall k \in \mathcal{K} \))

Let extreme points of \( \mathcal{B} \) be \( b^1, b^2, \ldots b^r \).
Let \( \mathcal{B} \) be a \( r \times n \) matrix whose rows are \( b^1, b^2, \ldots b^r \). Let \( \mathcal{A} = \{ y : y \geq 0, By \geq (i) \} \)
Then:
(i) \( \mathcal{A} \oplus = \hat{\mathcal{B}} \)
(ii) \( \mathcal{B} \) is proper, assurance \( \mathcal{A} \) is proper.
    ext pts of \( \mathcal{A} \) are rows of \( \mathcal{B} \).
(iii) \( \hat{A} = \mathcal{B} \).
    (i.e., Blocker of blocker is the original system)
\[
\hat{A} = \{ x : x \geq 0, x^T y \geq 1 \forall y \in \mathcal{A} \}
\]
We can extend all the results in Blocking (6) Systems to Blocking Polyhedra.