Matroids:

These are combinatorial structures that abstract the properties of linear dependence in vector spaces and related properties in graphs. There is relation to many areas including projective geometry, incidence geometries, in addition to matrices & graphs.

One of the first papers (fundamental) is due to Hassler Whitney. There are also deep results due to William Tutte, H. Crapo and G.C. Rota among others.

J. Edmonds was the first look at this from optimization point of view. A result due to David Gale is also important.

This has since been generalized to Δ matroids (also called Pseudo-matroids) and oriented matroids. We cover only the basics and focus on optimization rather than structures

There are many (fartoomany) ways of defining a matroid. We will show suse a few.

Independence

This is a direct generalization from linear algebra.

Def: Let E be a finite set [Some people allow E to be infinite → but we restrict to finite sets]

Let $\mathcal{F}$ be a family of subsets of E such that

$F \in \mathcal{F}, G \subset F \Rightarrow G \in \mathcal{F}$. Then $(E, \mathcal{F})$ is

Called an independence system

[almost opposite of a clutter !!!]

Examples: 1.) E is a finite set of vectors

$\mathcal{F}$ is a family of subsets of E that are linearly independent.

2). ~~G = [V, E]~~ G is an undirected graph.

E = set of vertices of G

$\mathcal{F}$ is a family of subsets of V that form an independent set

3). E set of edges of a graph G (undir)

$\mathcal{F}$ is a family of matchings in G.

<u>Def</u>: (Independence definition #1 of a matroid)

Let $(E, \mathcal{F})$ be an independence system satisfying:

$$\left[F \in \mathcal{F}, G \in \mathcal{F}, |G| > |F|\right]$$
$$\Rightarrow \exists\, g \in G - F \text{ such that } F \cup \{g\} \in \mathcal{F}.$$

is called a <u>Matroid</u>. Members of $\mathcal{F}$ are called independent sets.

<u>Example</u>:

1. E is the set of edges of an undir. graph G. $\mathcal{F}$ is a family of subsets of E such that the subgraph on sets $F \in \mathcal{F}$ are <u>cycle free</u>. Then $(E, \mathcal{F})$ is a <u>forest</u> matroid.
2. E is the set of columns of a matrix A^e. $\mathcal{F}$ is ~~sa~~ family of subsets of columns that are linearly independent. Then $(E, \mathcal{F})$ is a matroid. Here we could have elements of A come from any <u>field</u>.

Called Matric Matroid

If a matroid corresponds to this example in some field, M is said to be <u>representable</u> in that field.

<u>Definition</u> : Maximal independent ~~sets~~ subsets of a set $A \subseteq E$, in a matroid $(E, \mathcal{I}) = M$ are called <u>Bases</u> of A. A basis of E is <u>simply called a basis of M</u>.

Whitney introduced this

<u>Def</u> [Independence definition #2 of a matroid]

If an independence system $(E, \mathcal{I})$ satisfies:

Edmonds prefers this

all maximal independent subsets of a set A have the same size (Called the <u>rank</u> of A denoted by $r(A)$)

then $(E, \mathcal{I})$ is a matroid.

<u>Definition</u> (<u>rank</u> definition #2 of a matroid)

Given a finite set E, a function $r: 2^E \to \mathbb{Z}_+$ Called the <u>rank function</u> of a matroid (E, r) if r satisfies:

(i) $r(\phi) = 0$

(ii) $S \subseteq T \subseteq E \Rightarrow r(T) \geq r(S)$: <u>monotonicity</u>

(iii) $r(S) \leq |S|$

(iv) $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$ $\forall S, T \subseteq E$

Submodularity

Edmonds way.

also Whitney

then (E, r) is a matroid.

<u>Def</u> (rank definition #1 of a matroid)

Given a finite set E, a function $r: 2^E \to \mathbb{Z}_+$ is called the rank function of a matroid if it satisfies:

Whitney

i) $r(\phi) = 0$

ii) $S \subseteq T \subseteq E \Rightarrow r(T) \geq r(S)$ Monotonicity

iii) $r(S \cup \{e\}) = r(S)$ or $r(S)+1$ $\forall S \subseteq E$.

iv) $[r(S \cup \{e\}) = r(S \cup \{f\}) = r(S)]$
$\Rightarrow r(S \cup \{e, f\}) = r(S)$ $\forall S, e, f$

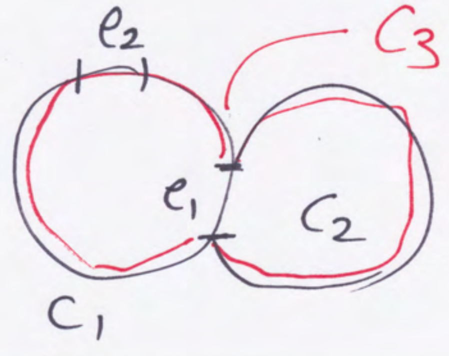
Then (E, r) is a matroid.

<u>Def</u>: (Circuit (cycle) definition #1)

Let E be a finite set; Let $\mathcal{C}$ be a clutter on E satisfying:

$$[C_1 \in \mathcal{C}, C_2 \in \mathcal{C}, e_1 \in C_1 \cap C_2, e_2 \in C_1 - C_2]$$
$$\Rightarrow [\exists C_3 \in \mathcal{C}; C_3 \subseteq C_1 \cup C_2; e_2 \in C_3, e_1 \notin C_3]$$

Whitney



Then $(E, \mathcal{C})$ is a matroid.

Def: (Circuit definition # 2)

$\mathcal{C}$ a clutter on E, satisfying

A. Lehman

$[C_1 \in \mathcal{C},\ C_2 \in \mathcal{C}\ ;\ e_1 \in C_1 \cap C_2]$

$\Rightarrow [\exists\ C_3 \in \mathcal{C},\ C_3 \subseteq C_1 \cup C_2 - \{e_1\}]$

Then $(E, \mathcal{C})$ is a matroid.

Bases: (Basis definition # 1)

Def: $\mathcal{B}$ is a clutter on E. Satisfying:

Whitney

$[B \in \mathcal{B},\ B' \in \mathcal{B},\ e' \in B' - B]$

$\Rightarrow [\exists\ e \in B - B'$ such that $B' - e' + e \in \mathcal{B}]$

then $(E, \mathcal{B})$ is a matroid

Def (Basis definition # 2]

$\mathcal{B}$ is a clutter on E satisfying

$[B \in \mathcal{B};\ B' \in \mathcal{B},\ e' \in B' - B] \Rightarrow$

$[\exists\ e \in B - B'$ such that $(B - e + e') \in \mathcal{B}]$

Then $(E, \mathcal{B})$ is a matroid.

There are many other ways to define a matroid: Now to some Connections and Examples:

Given $M=(E, \mathcal{F})$ (ind. set ~~def~~ definition)

→ (i) $r(S) = \max_{\substack{T \subseteq S \\ T \in \mathcal{F}}} |T|$..

(ii) $\mathcal{C}$ is the collection of minimal dependent subsets of E.

(iii) $\mathcal{B}$: maximal independent sets in $\mathcal{F}$.

Given $M = (E, r)$

(i) $\mathcal{F}$ is collection of subsets F with
$|F| = r(F)$

(ii) $\mathcal{B}$ and $\mathcal{C}$ are defined using $\mathcal{F}$.

Given $M = (E, \mathcal{C})$

(i) $\mathcal{F}$ is the collection of subsets F of E such that $\nexists\, c \in \mathcal{C},\ c \subseteq F$.

$\mathcal{B}$, r are defined based on $\mathcal{F}$.

Examples

1. Let E be a set of cardinality n. Let $\mathcal{F}$ be the family of all subsets of E of size $\leq k$. Then $(E, \mathcal{F})$ is a matroid. This matroid is called Cardinality matroid (or Uniform matroid). If $k = n$, it is called the <u>free</u> matroid.

2. Let $E = \bigcup_{i=1}^{p} E_i$, where E_i are pairwise disjoint.

   ~~Let $\mathcal{F} = \bigcup_{i=1}^{p} \mathcal{F}_i$~~

   Let $M_i = [E_i, \mathcal{F}_i]$ be matroids $i = 1 \cdots p$

   Let $\mathcal{F} = \left\{ F \subseteq E \, ; \, F = \bigcup_{i=1}^{p} F_i, \; F_i \in \mathcal{F}_i \; i = 1 \cdots p \right\}$

   $(E, \mathcal{F})$ is a matroid. This matroid is the Union of matroids (also called partition matroid

3. E is a (finite) set of vectors in a vector space. $\mathcal{F}$ = family of subsets of E that are lin. indep. $(E, \mathcal{F})$ is a matroid. Matroids of this type are called <u>linear</u> (or representable matroids)

4. E is the set of edges of a directed graph. $\mathcal{F}$ is family of subsets of E such that no more than an edge enters any node.

5. E : set of edges of an undir. graph.
$\mathcal{J}$: subset of E that have no more than 1-cycle.
$(E, \mathcal{J})$ is a matroid; called 1-forest.

6. Making matroids from other matroids
Let $(E, \mathcal{F}) = (E, \mathcal{B}) = (E, r) = (E, \mathcal{C})$ be a matroid M.

(i) Let $\mathcal{F}_k = \{F : F \in \mathcal{F}, |F| \leq k\}$. Then $M_k = (E, \mathcal{J}_k)$ is a matroid. This operation is known as k-Truncation of M.

(ii) Let $\hat{\mathcal{B}} = \{E-B : B \in \mathcal{B}\}$. Then $(E, \hat{\mathcal{B}}) = M^*$ matroid is called the dual of matroid M

(iii) Let $\mathcal{F} \setminus e = \{F : F \in \mathcal{F}, e \notin F\}$.
Then $\{E - e ; \mathcal{F} \setminus e\}$ is a matroid denoted by $M \setminus e$. This a matroid obtained from M by deleting an "edge" $e \in E$

(iv) Given M, $M/e = (M^* \setminus e)^*$. This matroid operation is called Contraction
(Very similar to contracting an edge in graphs.)

(v) For disjoint subsets $A, B \subseteq E$, deleting A and contracting B leads to a minor of M

Somewhat like Subgraphs

7. Transversal Matroids:

a) Let $E = \{e_1, e_2, \ldots, e_n\}$ be a finite set

Let $Q = \{q_1, q_2 \ldots, q_m\}$ be a family of (not necessarily distinct) subsets of E.

For example E might be a set of machines; there are m jobs to be performed and $q_i \subseteq E$ represent the subset of machines that can do job j.

Def: $T \subseteq E$; $T = \{e_{j_1}, e_{j_2} \ldots, e_{j_t}\}$ is called a partial transversal (partial assignment) of size t of Q if the elements of T are distinct and $e_{j_k} \in q_{i_k}$, $k = 1 \ldots t$ where $i_1, \ldots i_t$ are distinct indices.

The set T is called a transversal or (a system of distinct representatives) if $t = m$.

Theorem: Let E, Q be as above. Then $M_a = [E, \mathcal{F}]$ where $\mathcal{F}$ is the family of partial transversals is a matroid (Called Transversal Matroid]

Theorem (Contd)

$M_b = [Q, \mathcal{G}]$ where $\mathcal{G}$ is the family of Subsets of Q that have a (complete) transversal is also a matroid.

8. Let P be a set of points in the plane (or R^n). Let $G = [P; E]$ be a graph where E Contains all pairwise connections in P. Consider $[E, \mathcal{F}]$ where $F \in \mathcal{F}$ if no two members of F are parallel. $(E, \mathcal{F})$ is a partition matroid.

Examples of submodular functions arising in The context of matroids

1. Let $G = [V; E]$ be a directed graph. Let $r(X)$ be # of edges entering $X \subseteq V$
   $r(X)$ is submodular
2. Let $G = [V; E]$ be an undir. graph. Let $r(X) = |V| - k(V, X)$ ~~where~~ $X \subseteq E$
   where k is # of Connected Components
   r: is Submodular