Matroids:
These are combinatorial structures that abstract the properties of linear dependence in vector spaces and related properties in graphs. There is relation to many areas including projective geometry, incidence geometry, and addition to matrices & graphs.

One of the first papers (fundamental) is due to Hassler Whitney. There are also deep results due to William Tutte, H. Crapo and G.C. Rota among others.

J. Edmonds was the first look at this from optimization point of view. A result due to David Gale is also important.

This has since been generalized to Delta matroids (also called pseudo-matroids) and Oriented matroids. We cover only the basics and focus on optimization rather than structure.
There are many (fantosmany) ways of defining a matroid. We will show just a few.

**Independence**

This is a direct generalization from linear algebra.

**Def:** Let $E$ be a finite set [Some people allow $E$ to be infinite — but we restrict to finite sets] let $F$ be a family of subsets of $E$ such that $F \in F$, $G \subseteq F \Rightarrow G \in F$. Then $(E, F)$ is called an **independence system**

[almost opposite of a clutter! ! ! !]

**Example:** 1) $E$ is a finite set of vectors

$F$ is a family of subsets of $E$ that are linearly independent.

2) $G = (V, E)$ is an undirected graph.

$E$ = set of vertices of $G$

$F$ is a family of subsets of $V$ that form an independent set.

3) $E$ set of edges of a graph (undir).

$F$ is a family of matchings in $G$. 
Def. (Independence definition #1 of a matroid)

Let \((E, F)\) be an independence system satisfying:

\[
F \in \mathcal{F}, G \supset F, |G| > |F|
\]

\(\Rightarrow \exists g \in G - F \text{ such that } F \cup \{g\} \in \mathcal{F}.
\)

is called a **Matroid**. Members of \(\mathcal{F}\) are called independent sets.

**Example:**

1. \(E\) is the set of edges of an undirected graph \(G\). \(\mathcal{F}\) is a family of subsets of \(E\) such that the subgraph on sets \(F \in \mathcal{F}\) are cycle free. Then \((E, \mathcal{F})\) is a forest matroid.

2. \(E\) is the set of columns of a matrix \(A\). \(\mathcal{F}\) is a family of subsets of columns that are linearly independent. Then \((E, \mathcal{F})\) is a matroid. Here we could have elements of \(A\) come from any field of a matroid corresponds to this example in some field, \(M\) is said to be representable in that field.
Definition: Maximal independent sets $\mathcal{S}$ of a set $A \subseteq E$, in a matroid $(E, \mathcal{I}) = M$ are called **Bases** of $A$. A basis of $E$ is simply called a basis of $M$.

Definition [Independence definition #2 of a matroid]

If an independence system $(E, \mathcal{I})$ satisfies:

1. All maximal independent subsets of a set $A$ have the same size (Called the **rank** of $A$ denoted by $r(A)$)

then $(E, \mathcal{I})$ is a matroid.

Definition (rank definition of a matroid)

Given a finite set $E$, a function $r : 2^E \to \mathbb{Z}_+$ called the **rank function** of a matroid $(E, r)$ if $r$ satisfies:

1. $r(\emptyset) = 0$
2. $\forall S \subseteq E \Rightarrow r(T) \geq r(S) :$ **monotonicity**
3. $r(S) \leq |S|$
4. $r(S \cup T) + r(S \cap T) \leq r(S) + r(T) \forall S, T \subseteq E$ **submodularity**

then $(E, r)$ is a matroid.
Def (rank definition #1 of a matroid)

Given a finite set \( E \), a function \( r: 2^E \to \mathbb{Z}_4 \) is called the rank function of a matroid if it satisfies:

(i) \( r(\emptyset) = 0 \)
(ii) \( S \subseteq T \subseteq E \Rightarrow r(T) \geq r(S) \)  Monotonically
(iii) \( r(S \cup \{e\}) = r(S) \) or \( r(S) + 1 \)  \( \forall S \subseteq E \).
(iv) \[ r(S \cup \{e,f\}) = r(S) \] \( \Rightarrow r(S \cup \{e,f\}) = r(S) \)  \( \forall S, e, f \)

Then \( (E, r) \) is a matroid.

Def: (Circuit (cycle) definition #1)

Let \( E \) be a finite set; let \( \mathcal{C} \) be a clutter on \( E \) satisfying:

\[ C_1 \in \mathcal{C}, C_2 \in \mathcal{C}, e, e \in C_1 \cap C_2 \]
\[ \Rightarrow [\exists C_3 \in \mathcal{C}; C_3 \subseteq C_1 \cup C_2; e_2 \in C_3, e \notin C_3] \]

Then \( (E, \mathcal{C}) \) is a matroid.
Def: (Circuit definition # 2)

Let $\mathcal{B}$ be a clutter on $E$, satisfying

$$[c_1 \in \mathcal{B}, c_2 \in \mathcal{B} ; e_1 \in c_1 \cap c_2]$$

$$\Rightarrow [\exists c_3 \in \mathcal{B}, c_3 \subseteq c_1 \cup c_2 - \{e_1, e_2\}]$$

Then $(E, \mathcal{B})$ is a matroid.

Base: (Basis definition # 1)

Def: $\mathcal{B}$ is a clutter on $E$, satisfying:

Whitney

$$[B \in \mathcal{B}, B' \in \mathcal{B}, e \in B' - B]$$

$$\Rightarrow [\exists e \in B - B' \text{ such that } B' - e + e \in \mathcal{B}]$$

Then $(E, \mathcal{B})$ is a matroid.

Def: (Basis definition # 2)

$\mathcal{B}$ is a clutter on $E$, satisfying

$$[B \in \mathcal{B}, B' \in \mathcal{B}, e \in B' - B] \Rightarrow$$

$$[\exists e \in B - B' \text{ such that } (B - e + e') \in \mathcal{B}]$$

Then $(E, \mathcal{B})$ is a matroid.
There are many other ways to define a matroid. Now to some examples:

Given $M = (E, F)$ (ind. set def. definition)

$$r(S) = \max_{T \subseteq S} \left| \frac{1}{T \cap F} \right|$$

(i) $G$ is the collection of minimal dependent subsets of $E$.

(ii) $B$ is the collection of maximal independent sets in $F$.

Given $M = (E, r)$

(i) $F$ is collection of subsets $F$ with
$$|F| = r(F)$$

(ii) $B$ and $G$ are defined using $F$.

Given $M = (E, G)$

(i) $F$ is the collection of subsets $F$ of $E$ such that $F \subseteq C \subseteq E$, $C \subseteq F$.

$B$, $r$ are defined based on $F$. 

Examples:

1. Let $E$ be a set of cardinality $n$. Let $F$ be the family of all subsets of $E$ of size $\leq k$. Then $(E, F)$ is a matroid. This matroid is called the \textit{cardinality matroid} (or \textit{uniform matroid}). If $k = n$, it is called the \textit{free matroid}.

2. Let $E = \bigcup_{i=1}^{p} E_i$, where $E_i$ are pairwise disjoint.

   Let $F_i = \bigcup_{i=1}^{p} F_i$

   Let $M_i = [E_i, F_i]$ be matroids $i = 1 \ldots p$

   Let $F = \{ F \subseteq E ; F = \bigcup_{i=1}^{p} F_i, F \subseteq F_i, i = 1 \ldots p \}$

   $(E, F)$ is a matroid. This matroid is the union of matroids (also called partition matroid).

3. $E$ is a (finite) set of vectors in a vector space. $F$ is family of subsets of $E$ that are lin. indep. $(E, F)$ is a matroid. Matroids of this type are called \textit{linear} (or \textit{representable} matroids).

4. $E$ is the set of edges of a directed graph. $F$ is family of subsets of $E$ such that no more than one edge enters any node.
5. $E$: set of edges of an undir. graph.
   $F$: Subset of $E$ that have no more than 1 cycle.
   $(E,F)$ is a matroid; called 1-forest.

6. Making matroids from other matroids,
   i) Let $(E,F) = (E, \emptyset) = (E, r) = (E, G)$ be a matroid $M$.
   
   ii) Let $F_k = \{ F : F \in F, |F| \leq k \}$, Then $M_k = (E, F_k)$ is a matroid. This operation is known as k-Truncation of $M$.
   
   iii) Let $\hat{B} = \{ B : B \in B \}$. Then $(E, \hat{B}) = M^*$ matroid is called the dual of matroid $M$.
   
   (vi) Let $F \setminus e = \{ F : F \in F, e \not\in F \}$. Then $\{ F \setminus e : F \in F \}$ is a matroid denoted by $M \setminus e$. This is a matroid obtained from $M$ by deleting an "edge" $eeF$.

(IV) Given $M$, $M/e = (M \setminus e)^*$. This matroid operation is called Contraction (Very similar to contracting an edge in graphs).

(V) For disjoint subsets $A, B \subseteq E$, deleting $A$ and Contracting $B$ leads to a minor of $M$. 
7. Transversal Matroid

(a) Let \( E = \{ e_1, e_2, \ldots, e_n \} \) be a finite set.

Let \( \mathcal{Q} = \{ q_1, q_2, \ldots, q_m \} \) be a family of

(not necessarily distinct) subsets of \( E \). For example, \( E \) might be a set of machines, then there are \( m \) jobs to be performed and \( q_i \subseteq E \)

represents the set of machines that can do job \( j \).

**Def:** \( T \subseteq E \); \( T = \{ e_{j_1}, e_{j_2}, \ldots, e_{j_t} \} \) is called

a **partial transversal** (partial assignment) of size \( t \) of \( \mathcal{Q} \) if the elements of \( T \)

are distinct and \( e_{j_k} \in q_{i_k}, k = 1, \ldots, t \)

where \( i_1, \ldots, i_t \) are distinct indices.

The set \( T \) is called a **transversal** or (a system of distinct representatives) if \( t = m \).

**Theorem:** Let \( E, \mathcal{Q} \) be as above. Then \( M = [E, \mathcal{F}] \) where \( \mathcal{F} \) is the family of partial transversals is a matroid (called Transversal Matroid)
Theorem (Contd)

$M_b = [Q, G]$ where $G$ is the family of subsets of $Q$ that have a (complete) transversal is also a matroid.

8. Let $P$ be a set of points in the plane (or $\mathbb{R}^n$). Let $G = [P; E]$ be a graph where $E$ contains all pairwise connections in $P$. Consider $[E, F]$ where $F$ is a family of sets of $F$ if no two members of $F$ are parallel. $(E, F)$ is a partial matroid.

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Example of submodular functions arising in the context of matroids,

1. Let $G = [V; E]$ be a directed graph. Let $\sigma(X)$ be the number of edges entering $X \subseteq V$.

$\tau(X)$ is submodular.

2. Let $G = [V; E]$ be an undirected graph. Let

$\tau(X) = |V| - k(V, X)$ where $X \subseteq E$ and $k$ is the number of connected components.

$\tau$ is submodular.