CS6381- Lecture # 26 10.31.2020 (1)

Showing a system is a matroid

1. Cardinality Matroid

$(E, \mathcal{F})$; $\mathcal{F} = \{F \subseteq E, |F| \leq k\}$.

If $F \in \mathcal{F}$, $G \subseteq F$; $|F| \leq k$, $\therefore |G| \leq k$ $\therefore G \in \mathcal{F}$.

$(E, \mathcal{F})$ is an independence system.

If $F \in \mathcal{F}$, $G \in \mathcal{F}$, $|G| > |F|$, then $|F| \leq k-1$.

$\therefore \exists\, g \in G, g \notin F$, $|\{F \cup g\}| \leq k$ $\therefore F \cup \{g\} \in \mathcal{F}$.

$\therefore (E, \mathcal{F})$ is a matroid.

2. Partition Matroid (disjoint Union)

$E = \dot{\bigcup}_{i=1}^{p} E_i$, E_i are pairwise disjoint

$M_i = (E, \mathcal{F}_i)$ are matroids ; $i = 1 \cdots p$

$\mathcal{F} = \{F \subseteq E : F = \dot{\bigcup}_{i=1}^{p} F_i, F_i \in \mathcal{F}_i\}$

Clearly this is an independence system since $(E, \mathcal{F}_i)$ are ind. systems $i = 1, \cdots p$

Suppose $\dot{\bigcup}_{i=1}^{p} F_i = F \in \mathcal{F}$, where $F_i \in \mathcal{F}_i$ $i = 1 \cdots p$

Suppose $\dot{\bigcup}_{i=1}^{p} G_i = G \in \mathcal{F}$ where $G_i \in \mathcal{F}_i$ $i = 1 \cdots p$

Suppose $|G| > |F|$ Then for some i, $|G_i| > |F_i|$.

$\therefore\ \exists\, g_i \in G_i - F_i$ such that $F_i \cup g_i \in \mathcal{J}_i$

[Note $g_i \in E_i$; hence $g_i \notin E_j$ $j \neq i$]

$\therefore\ F \cup \{g_i\} \in \mathcal{J}$ $\therefore\ \exists\, g_i \in G$ such that $F \cup \{g_i\} \in \mathcal{J}$

$\therefore (E, \mathcal{J})$ is a matroid.

4. E is the set of edges of a dir. graph
$\mathcal{J}$ is the family of subsets of E such that no more than one edge enters any node
Want to show $(E, \mathcal{J})$ is a matroid.
"Clearly" $(E, \mathcal{J})$ is an ind. system
Let $E = \bigcup E_i$ where E_i are edges entering node i.

Let $\mathcal{J}_i = \{F_i \subseteq E_i : |F_i| \leq 1\}$

then $(E_i, \mathcal{J}_i) = M_i$ is a Cardinality matroid

$(E, \mathcal{J}) = M$ is union M_i. Hence $(E, \mathcal{J})$ is a matroid.

5. 1-forest ; $(E, \mathcal{J})$ is clearly an ind. system
Forests

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③ Forest Matroid

$G = [V; E]$ undir. graph

Let $\mathcal{F} : \{ F \subseteq E ;$ no cycles in $(V, F) \}$

Claim $(E, \mathcal{F})$ is a matroid.

Let $F \in \mathcal{F}$, $G \in \mathcal{F}$, $|G| > |F|$.

Look at connected components of $(V; F)$

$C_1, \cdots C_k$ Each is a tree

# of connected components of $(V; F)$

$$= |V| - |F|$$

Since there are no cycles in F.

Since $G \in \mathcal{F}$, no cycles in (V, G) # of connected components of $G = |V| - |G| < |V| - |F|$

Since $|G| > |F|$

$\therefore$ Some edge $g \in G$ must connect two components in $[V; F]$; & This edge can be added to F without creating any cycle. $\therefore F \cup \{g\} \in \mathcal{F}$.

$\therefore (E, \mathcal{F})$ is a matroid.

1-forest:

Let $\{E, \mathcal{F}\}$ be such that $F \in \mathcal{F}$ has $(V; F)$ with not more than 1-cycle.

Claim $(E, \mathcal{F})$ is a matroid

Pf Clearly $(E, \mathcal{F})$ is an ind. system.

Let $F \in \mathcal{F}$, $G \in \mathcal{F}$, $|G| > |F|$.

Case 1: F has no cycle.

~~If # of connected compon~~

Since $|G| > |F|$, $\exists g \in G$, $g \notin F$.

$F \cup \{g\}$ Contains at most one cycle.

$\therefore F \in \mathcal{F}$.

Case 2: F has a cycle. In this case,

\# of Connected components in $(V; F)$

$= |V| - |F| + 1$

$> |V| - |G|$

$\therefore \exists g \in G$ such that g connects two components of $(V; F)$.

$\therefore F \cup \{g\}$ Contains only one cycle

$\therefore (E, \mathcal{F})$ is a matroid

6. ii) k-truncation of a matroid.

Let $(E, \mathcal{F})$ be a matroid. Let

$$\mathcal{F}_k = \{ F \subseteq \mathcal{F} : |F| \le k \}$$

Claim $(E, \mathcal{F}_k)$ is a matroid

Pf Clearly $(E, \mathcal{F}_k)$ is an ind. system since $(E, \mathcal{F})$ is one.

Let $F \in \mathcal{F}_k$, $G \in \mathcal{F}_k$, $|G| > |F|$

$$\Downarrow$$

$F \in \mathcal{F}$, $G \in \mathcal{F} \Rightarrow \exists g \in G$ such that $F \cup \{g\} \in \mathcal{F}$

Since $(E, \mathcal{F})$ is a matroid.

Since $|G| > |F|$, $|G| \le k$,

$|F| \le k-1$

$\therefore |F \cup \{g\}| \le k$

$\therefore F \cup \{g\} \in \mathcal{F}_k$ $\therefore (E, \mathcal{F}_k)$ is a matroid

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Dual Matroid $\rightarrow$ next page

6 (ii) Dual Matroid:

Let $(E, \mathcal{B})$ be a matroid.

Claim: $(E, \hat{\mathcal{B}})$ is a matroid

where $\hat{\mathcal{B}} = \{ \hat{B} : \hat{B} = E - B, \text{ for some } B \in \mathcal{B} \}$

$\hat{\mathcal{B}}$ is a clutter:

Pf: If $\hat{B}_1 \in \hat{\mathcal{B}}$, $\hat{B}_2 \in \mathcal{B}$, $\hat{B}_1 \subsetneq \hat{B}_2$

$\hat{B}_1 = E - B_1$, $B_1 \in \mathcal{B}$

$\hat{B}_2 = E - B_2$, $B_2 \in \mathcal{B}$

$\hat{B}_1 \subsetneq \hat{B}_2 \Rightarrow B_2 \subsetneq B_1$.

But $\mathcal{B}$ is a clutter. Hence the result.

Claim. $(E, \hat{\mathcal{B}})$ is a matroid

Need to show $\hat{B}_1 \in \hat{\mathcal{B}}$, $\hat{B}_2 \in \hat{\mathcal{B}}$, $e' \in \hat{B}_2 - \hat{B}_1$

then $\exists$ $e \in \hat{B}_1 - \hat{B}_2$ such that $\hat{B}_2 - e' + e \in \hat{\mathcal{B}}$.

Equivalent to showing $E - \{ \hat{B}_2 - e' + e \} \in \mathcal{B}$

Let $E - \hat{B}_1 = B_1 \in \mathcal{B}$, $E - \hat{B}_2 = B_2 \in \mathcal{B}$,

$e' \in E - B_1$, $e' \notin E - B_2$;

$(E, \mathcal{B})$ is a matroid.

$[B \in \mathcal{B}, B' \in \mathcal{B} ; e' \in B' - B]$

$\Rightarrow [\exists\ e \in B - B'$ such that $B' - e' + e \in \mathcal{B}]$

$[\hat{B} = E - B \in \hat{\mathcal{B}} ; \hat{B}' = E - B' \in \hat{\mathcal{B}} ; e' \in \{\hat{B} - \hat{B}'$ $]$

$\Rightarrow [\exists\ e \in \hat{B}' - \hat{B}$ such that $(E - \hat{B}') - e' + e \in \mathcal{B}$

$\Updownarrow$

$\cancel{\hat{B} - e' + e}$

$\hat{B}' + e' - e \in \hat{\mathcal{B}}]$

By def #2, $(E, \hat{\mathcal{B}})$ is a matroid.

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6 iii) & (iv) follow from above.

7(a) Now we turn to <u>transversal matroids</u>

$\underline{E} = \{e_1, e_2, \ldots e_n\}$

$\mathcal{Q} = \{q_1, q_2, \ldots ; q_m\}$ (not necessarily distinct

$\mathcal{J}_a$: family of partial transversals

$M = [E, \mathcal{J}_a]$: matroid

$T \in \mathcal{J}_a$:

$$T: \quad \begin{array}{cccc} e_{j_1}, & e_{j_2}, & \cdots & e_{j_t} \\ \cap & \cap & \cdots & \cap \\ q_{i_1} & q_{i_2} & & q_{i_t} \end{array}$$

distinct j

distinct i

Clearly $(E, \mathcal{J}_a)$ is an independence system.

Let $F, G \in \mathcal{J}_a$, $|G| > |F|$ Say $\underline{|G| = |F| + 1}$ (rest will follow from this)

$$F = \{ e_{j_1}, \cdots e_{j_t} \}$$

$$G = \{ e_{j'_1}, \cdots e_{j'_{t+1}} \}$$

Then general situation:

$$\overbrace{\begin{array}{ccc|ccc|} e_{j_1} & \cdots & e_{j_r} & e_{j_{r+1}} & \cdots & e_{j_t} \\ \cap & & \cap & \cap & & \\ q_{i_1} & & q_{i_r} & q_{i_{r+1}} & \cdots & q_{i_t} \end{array}}^{F}$$

$$\downarrow \qquad \underbrace{\begin{array}{|ccc|} e_{j'_{r+1}} & \cdots & e_{j'_{t+1}} \\ \cap & & \cap \\ q_{i'_{r+1}} & \cdots & q_{i'_{t+1}} \end{array}}_{G}$$

Since there is one more pair for G, $\exists\ q_{i'}$ not (9)

in the pairs corresponding to F. If the corresp.

$e_{j'} \notin F$, then we add this pair to pairs in F

and $\exists\ g = e_{j'} \in G - F$ such that $F \cup \{g\}$ is

also a partial transversal. and we are done.

If $e_{j'} = e_{j_k} \in F$, then we know $k > r$

Without loss let $e_{j'} = e_{j_{r+1}}$ and replace

q_{i+1} in F side with $q_{i'}$. This moves

the partition one index to the right.

Repeating this we get the result.

Hence $(E, \mathcal{I}_a)$ is a matroid.

7(b) Show $M_b = (E, \mathcal{I}_b)$ is a matro is similar

Theorem (Nash-Williams)

Let $M = [E, \mathcal{J}]$ be a matroid.

$E \xrightarrow[h]{} \hat{E}$: be mapping of $E \to \hat{E}$.

$h(F) : \{\hat{e} : \hat{e} = h(f)\ f \in F\}$

$\hat{\mathcal{J}} = \{\hat{F} : \hat{F} = h(F) \text{ some } F \in \mathcal{J}\}$

Then $[\hat{E}, \hat{\mathcal{J}}]$ is a matroid.

Pf "Clearly" $(\hat{E}, \hat{\mathcal{J}})$ is an ind. system.

Suppose $\hat{F} \in \hat{\mathcal{J}}$, $\hat{G} \in \hat{\mathcal{J}}$, $|\hat{G}| > |\hat{F}|$.

Let $\hat{F} = h(F)$; $\hat{G} = h(G)$; $F \in \mathcal{J}, G \in \mathcal{J}$

Select F so that $|F| = |\hat{F}|$, ← This is always possible.

G .. $|G| = |\hat{G}|$

(since $h^{-1}(\hat{e}) \cap h^{-1}(\hat{f}) = \phi$ for $\hat{e} \neq \hat{f}$)

$\therefore |G| > |F|$; $\therefore \exists\, g \in G - F$ such that $F \cup \{g\} \in \mathcal{J}$.

$\left.\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right\} F \cap G$

$\in F \left\{\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}\right. \quad \left.\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}\right\} \in G \quad h(f) = h(g)\ \ f \in F,\ g \in G$

$\in F \left\{\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}\right. \quad \left.\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right\} \in G \quad h(f_i) \neq h(g_j)\ \forall i, j$

If g is in the third block, we are done.

(We can add $h(g)$ to $\hat{F}$ and still be in $\hat{\mathcal{F}}$.

$\in h(G) = \hat{G}$

If g is in the second part, replace $f \in F$ by g such that $h(f) = h(g)$ and this now goes into the first part

Thus, after a finite number of steps, the second part is empty and at this point since $|\hat{G}| > |\hat{F}|$, there must be a g in the third part and $\hat{F} \cup \{h(g)\} \in \hat{\mathcal{F}}$.

Hence the result.

<u>Corollary</u>: Let $M_i = (E, \mathcal{F}_i)$ $i = 1 \dots p$.

Let $M = [E, \mathcal{F}]$ where

$$\mathcal{F} = \{F : F = \bigcup_{i=1}^{p} F_i \;, F_i \in \mathcal{F}_i \quad i = 1 \dots p\}.$$

Then M is a matroid (called the union of M_i

<u>Pf</u>: Make p copies of E : $E_1, \dots E_p$.

Let $\tilde{M}_i = [E_i, \mathcal{F}_i]$ $i = 1 \dots p$ (all matroids)

Let $\tilde{M} = [\bigcup_{i=1}^{p} E_i, \Pi \mathcal{F}_i \quad i = 1 \dots p]$

($\tilde{M}$ is a matroid : disjoint union)

Let $h : \bigcup_{i=1}^{p} E_i \to E$ defined by

$h(e_i) = e \quad i = 1 \dots p$ | e_i are all copies of e.

$\tilde{M} \to M$.

alternate

Proof of dual matroid: (E. Lawler) (12)

Let $M = (E, \mathcal{F})$. Let $\mathcal{F}^* = \{ F \subseteq E : \exists B \in \mathcal{B}, B \subseteq E - F \}$
$\quad = (E, \mathcal{B})$

Then $(E, \mathcal{F}^*) = M^*$ is a matroid (called the dual of M. $(E, \mathcal{F}^*) = (E, \mathcal{B}^*)$ where
$\mathcal{B}^* = \{ B^* \subseteq E : B^* = E - B$ for some $B \in \mathcal{B} \}$

Proof via independence property.

Clearly $(E, \mathcal{F}^*)$ is an ind. system.

Let $F^*, G^* \in \mathcal{F}^*$, $\underset{p+1}{|G^*|} = \underset{p}{|F^*|} + 1$

Let $B_p, B_{p+1} \in \mathcal{B}$, $B_p \cap F^* = \phi = B_{p+1} \cap G^*$.

Case 1: $G^* - (F^* \cup B_p) \neq \phi$

Let $e \in G^* - (F^* \cup B_p)$ then since $F^* \cup \{e\}$ is disjoint from B_p, $F^* \cup \{e\} \in \mathcal{F}^*$ and we are done.

Case 2 Suppose $G^* - (F^* \cup B_p) = \phi$.
$\Downarrow$
$G^* \subseteq (F^* \cup B_p)$

See next page

Claim $B_{p+1} - (B_p \cup F^*)$ is non-empty.

Pf of claim: Suppose $B_{p+1} - (B_p \cup F^*) = \phi$.

$$\therefore B_{p+1} \subseteq B_p \cup F^*$$

$$\therefore B_{p+1} \cup G^* \subseteq B_p \cup F^*$$

and
$$|B_{p+1}| + \underset{|G^*|}{\underset{\parallel}{p+1}} \leq |B_p| + \underset{|F^*|}{\underset{\parallel}{p}}$$

a contradiction

Hence $B_{p+1} - (B_p \cup F^*)$ is non-empty.

Now select $e \in B_{p+1} - (B_p \cup F^*)$. $B_p \cup e$ contains a unique circuit in M. Let e' be any element of this circuit other than e. Then $B_p' = B_p + e - e' \in \mathcal{B}$.

And B_p' is disjoint from F^*.

If $G^* - (B_p' \cup F^*) \neq \phi$, Case 1 applies.

If $G^* - (B_p' \cup F^*) = \phi$, repeat the Case 2 argument with B_p' instead of B_p. Until case 1 occurs. This must happen since otherwise we will run out of elements in $B_{p+1} - (B_p \cup F^*)$ in a finite # of steps. Hence the result.

$$r^*(A) = |A| + r(E-A) - r(E) \quad \forall\, A \subseteq E. \quad (14)$$

<u>Pf</u>: $r^*(A)$ is determined by a basis of M with minimum # of elements in A. Max size of an ind. set of M disjoint from A is $r(E-A)$. Such a set is contained in a basis of M with $r(E)$ elements of which $r(E) - r(E-A)$ are contained in A. # of elements of A not contained in this basis is $|A| - (r(E) - r(E-A))$.