CS6381: Lecture 27 11-1-2020

Optimization on Matroids and Independence Systems

1. Given a finite set E with element weights $w(e), e \in E$, a family $\mathcal{F}$ of subsets of E find $F \in \mathcal{F}$ with

$$\max \sum_{e \in F} w(e) \quad \Big| \quad \text{or } \min \sum_{e \in F} w(e)$$

don't need this

Basis in (E, B) more approp

When does greedy algorithm work ?

Gale optimality

Let elements of E be ordered: $e_1 \geq e_2 \geq \cdots$

Let $F = \{f_1, f_2, \cdots f_k\}$, $G = \{g_1, g_2, \cdots g_l\}$

Where $F, G \in \mathcal{F}$. If $k \geq l$, and

$f_1 \geq g_1$; $f_2 \geq g_2, \cdots f_l \geq g_l$

then we say F is "better" than G.

Q: Is there an $F^* \in \mathcal{F}$ such that F^* is better than G $\forall G \in \mathcal{F}$. ?

Such a F^* is called Gale optimal solution

Greedy algorithm:

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Let E be a finite set; with ~~edge~~ element weights $w(e)$, $e \in E$. (2)

Let elements of E be arranged so that

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \cdots \geq w(e_n)$$

Where $|E| = n$. We may be interested in one of two problems assuming $(E, \mathcal{F})$ is a matroid. $(E, \mathcal{B})$

1. Find $\max\limits_{F \in \mathcal{F}} \sum\limits_{e \in F} w(e)$

2. Find $\max\limits_{B \in \mathcal{B}} \sum\limits_{e \in B} w(e)$

These are "slightly" different.

New matroid results

For (1) we ~~ignore~~ delete elements $e_{k+1}, \ldots e_n$. We need not (and will not) select any of them.

For (2) we may be "forced" to include some of these elements in order to get a $B \in \mathcal{B}$.

We can convert (2) to (1) as follows [1 → 2 is also possible a…

Since $|B|$ is same for all $B \in \mathcal{B}$ (matroid axioms), we can change $w(e)$ to $w'(e) = w(e) + K$, $K > 0$ large without affecting the result.

So we will look at (1) from now on.

Now opt. solution for (1) is a basis

Algorithm: [for (1)] [Greedy algorithm]

Process these elements up to e_k. Start with $F_0 = \phi$. $\in \mathcal{F}$

At any stage j, $(j \leq k)$ we have $F_j \in \mathcal{F}$ and are about to consider element e_{j+1}.

If $F_j \cup \{e_{j+1}\} \in \mathcal{F}$, let $F_{j+1} = F_j \cup \{e_{j+1}\}$.

If not let $F_{j+1} = F_j$ and proceed to process e_{j+2}.

Theorem (Gale): Let $M = (E, \mathcal{F})$ be a matroid

Let elements ~~be~~ of E be ordered: $e_1 \geq e_2 \geq \dots \geq e_n$ (assume $w(e) \geq 0 \ \forall e \in E$)

$\exists$ a Gale optimal member in $\mathcal{F}$.

Pf: Apply greedy algorithm to determine a F^* the lexico-maximum member of $\mathcal{F}$. Claim F^* is Gale optimal.

Pf of claim: Let $F^* = \{f_1, f_2, \dots f_m\}$ $f_i \in E$
with $f_1 \geq f_2 \geq \dots \geq f_m$

Let $G \in \mathcal{F}$ be any member of $\mathcal{F}$ &

$$G = \{g_1, g_2, \dots g_n\}$$
$$g_1 \geq g_2 \dots \geq g_n$$

Suppose $g_k > f_k$ (with k min)

$\Rightarrow \quad g_1, g_2 \ldots, g_{k-1} > f_k$

Consider $F^*_{k-1} = \{f_1, \ldots f_{k-1}\} \in \mathcal{J}$

$G_k = \{g_1, \ldots g_k\} \in \mathcal{J}$.

$|G_k| > |F^*_{k-1}| \therefore \exists\, g_i \in G_k - F^*_{k-1}$ such that $F^*_{k-1} \cup \{g_i\} \in \mathcal{J}$.

$F^*_{k-1} \cup \{g_i\}$ is lexico-larger than F^*

— a contradiction.

Hence $g_i \le f_i \quad i = 1 \ldots n$. [implying $m \geqq n$].

$\therefore F^*$ is a (maximal) Gale opt. ~~set~~ member of $\mathcal{J}$ (one that is also maximum in size)

and hence a basis of M.

Hence greedy alg. $\max\limits_{F \in \mathcal{J}} \sum\limits_{e \in F} w(e)$ (assuming

$w(e) \geqq 0 \ \forall e \in E$; else truncate so $w(e) \geqq 0$

Now the converse.

<u>Theorem 2.</u> If $M = (E, \mathcal{I})$ is an ind. system such that greedy alg. correctly computes $\max_{F \in \mathcal{I}} \sum_{e \in F} w(e)$ for all w, then M is a <u>matroid</u>.

<u>Pf</u>: Need to show that if $F \in \mathcal{I}$, $G \in \mathcal{I}$, and $|G| > |F|$, $\exists\, g \in G - F$ such that $F \cup \{g\} \in \mathcal{I}$.

$$\text{Let } w(e) = \begin{cases} 1 & e \in F \\ 1-\epsilon & e \in G - F \\ 0 & e \notin G \cup F \end{cases}$$

The greedy algorithm <u>first</u> selects F; but

$$\sum_{e \in G} w(e) > |F| \text{ } \underline{\text{if}} \text{ } |G-F| \cdot (1-\varepsilon) > |F-G|$$

$$\text{i.e. } (1-\epsilon) > \frac{|F-G|}{|G-F|}$$

We can choose such an ϵ since $\dfrac{|F-G|}{|G-F|} < 1$

$\therefore$ alg. must continue to select more elements from $G-F$ after selecting F and we are done.

Edmonds' Proof Via LP duality.

1- Matroid optimization:

Ref.: Matroids and The Greedy Algorithm
Jack Edmonds, Mathematical Programming,
Vol 1 (1971) pp 127-136

Consider: $x_j \geqq 0 \quad j \in E = \{1, 2, \dots, n\}$

(P)

$y(s) \geqq 0$? $\quad \sum_{j \in S} x_j \leqq r(S) \quad \forall S \subseteq E.$ $\longrightarrow$ rank function of M.

$$\text{Max} \sum_{j=1}^{n} w_j x_j$$

Since $r\{j\} \in \{0, 1\}$, $0 \leq x_j \leq 1 \;\forall j = 1 \dots n$

Integer solutions to P (necessarily 0/1 solution) is an indicator function of $F \in \mathcal{F}$.

(D)

$$y(S) \geqq 0 \quad \forall S \subseteq E$$

$$\sum_{S: j \in S} y(S) \geqq w_j \qquad j = 1 \dots n$$

$$\text{Min} \sum_{S \subseteq E} r(S) \cdot y(S)$$

is the LP dual of (P).

Now Let elements of E be ordered so that

$$w_1 \geqq w_2 \geqq \cdots \geqq w_m \geqq 0 \geqq w_{m+1} \cdots \geqq w_n.$$

Let $E' = \{1, 2, \ldots m\} \subseteq E$.

Let $A_k = \{1, 2, \ldots, k\}$

Let $x^0 = (x_1^0, x_2^0, \ldots \quad)$

Where $x_1^0 = r(A_1)$; $x_j^0 = r(A_j) - r(A_{j-1})$ $j = 2, \ldots m$

and $x_j^0 = 0$ $j > m$.

[This is the solution produced by the greedy algorithm]

It is easy to verify that x^0 is the indicator vector of a set F which is a M-basis of E'.

Let $\{y^0(S)\}$ be defined as follows:

$$y^0(A_j) = \underbrace{w_j - w_{j+1}}_{\geqq 0 \,\to\, \text{by the ordering above.}} ; \; j = 1, \ldots m-1$$

$$y^0(A_m) = w_m$$

$$y^0(A) = 0 \quad \text{for all other } A \subseteq E.$$

Consider any element $j \in E$. If $j > m$

then $\sum_{j \in S} y^0(S) = 0 > w_j$

If $j \leq m$

$$\sum_{j \in A} y^0(\cancel{S}) = \sum_{k=j}^{m} y^0(A_k) =$$

$$= \cancel{\sum_{j=1}^{m-1} (w_j - w_{j+1})}$$

$$= \left(\sum_{k=j}^{m-1} (w_k - w_{k+1}) \right) + w_m$$

$$= w_j \geqq 0.$$

Hence y^0 is feasible to the dual.

Check to see (x^0, y^0) satisfy Complementary Slackness and hence optimal to (P) and (D) respectively.

[See page 279 of Cook, Cunningham, P and S]

Now to Matroid intersection and other problems