Optimization on Matroids and Independence Systems

1. Given a finite set $E$ with element weights $w(e), e \in E$, find $F \subseteq I$ with
   $\max \sum w(e)$
   $e \in F$
   or $\min \sum w(e)$
   $e \in F$

When does greedy algorithm work?

Gale optimality

Let elements of $E$ be ordered: $e_1 \geq e_2 \geq \ldots$

Let $F = \{f_1, f_2, \ldots, f_k\}$, $G = \{g_1, g_2, \ldots, g_k\}$

When $F, G \subseteq I$, if $k \geq l$, and

$f_1 \geq g_1$, $f_2 \geq g_2$, $\ldots$, $f_l \geq g_l$

Then we say $F$ is "better" than $G$.

Q: Is there an $F^\ast \in I$ such that $F^\ast$ is better than $G$ for all $G \in I$?

Such a $F^\ast$ is called Gale optimal solution.

Greedy algorithm:

See next page.
Let $E$ be a finite set, with edge weights $w(e) \in \mathbb{R}$. Let elements of $E$ be arranged so that

$$w(e_1) \geq w(e_2) \geq \ldots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \ldots \geq w(e_n)$$

where $|E| = n$. We may be interested in one of two problems, assuming $(E, F)$ is a matroid. 

1. Find $\max_{F \in \mathcal{F}} \sum_{e \in F} w(e)$
2. Find $\max_{B \in \mathcal{B}} \sum_{e \in B} w(e)$

These are slightly different.

For (1) we ignore elements $e_{k+1}, \ldots, e_n$. We need not (and will not) select any of them.

For (2) we may be "forced" to include some of these elements in order to get a $B \in \mathcal{B}$.

We can convert (2) to (1) as follows:

Since $|B|$ is same for all $B \in \mathcal{B}$ (matroid axiom), we can change $w(e)$ to $w(e) = w(e) + K$ for $K > 0$ large without affecting the result. So we will look at (1) from now on.

Now opt. solution $\{e_1, e_2, \ldots, e_k\}$ is a basis.
Algorithm: [full] [Greedy algorithm]

Process these elements up to \( E_k \). Start with \( F_0 = \emptyset \).

At any stage \( j \), \( j \leq k \) we have \( F_j \in \mathcal{F} \) and are about to consider element \( e_{j+1} \).

- If \( F_j \cup \{ e_{j+1} \} \in \mathcal{F} \), let \( F_{j+1} = F_j \cup \{ e_{j+1} \} \).
- Otherwise, let \( F_{j+1} = F_j \) and proceed to process \( e_{j+2} \).

**Theorem (Gale):** Let \( M = (E, \mathcal{F}) \) be a matroid.

Let elements of \( E \) be ordered: \( e_1 \geq e_2 \geq \cdots \geq e_n \) (assume \( w(e) > 0 \) for \( e \in E \)).

\( F \) a Gale optimal member in \( \mathcal{F} \).

**Pf:** Apply greedy algorithm to determine a

the lexicographically maximum member of \( \mathcal{F} \). Claim \( F^* \) is Gale optimal.

**Pf of claim:** Let \( F^* = \{ f_1, f_2, \ldots, f_m \} \) \( f_i \in E \)

with \( f_1 \geq f_2 \geq \cdots \geq f_m \)

Let \( G \in \mathcal{F} \) be any member of \( \mathcal{F} \).

\[ G = \{ g_1, g_2, \ldots, g_n \} \]

\[ g_1 \geq g_2 \geq \cdots \geq g_n \]
Suppose $g_k > f_k$ \((\text{with } k \text{ min})\)

\[ \Rightarrow g_1, g_2, \ldots, g_{k-1} > f_k \]

Consider $F^*_{k-1} = \{ f_1, \ldots, f_{k-1} \} \in \mathcal{F}$

$G_k = \{ g_1, \ldots, g_k \} \in \mathcal{F}$.

\[ |G_k| > |F^*_k| \quad \exists \quad g_i \in G_k - F^*_k \quad \text{such that} \quad F^*_{k-1} \cup \{ g_i \} \in \mathcal{F} \]

$F^*_{k-1} \cup \{ g_i \}$ is lexicographically larger than $F^*$

- a contradiction.

Hence $g_i \leq f_i \quad i = 1 \ldots n$ \([\text{implying } m \geq n]\).

\[ \Rightarrow F^* \quad \text{is a \textit{maximal} Gale opt. set member of } \mathcal{F} \]

\(\text{One that is also maximum in size}\)

And hence a basis of $M$.

Hence greedy alg. max $\sum_{e \in F} w(e)$ \((\text{assuming } F \in \mathcal{F})$

$w(e) \geq 0 \forall e \in \mathcal{E}$; \underline{else truncate} so $w(e) \leq 0$
Now the converse.

**Theorem 2.** If \( M = (E, G) \) is an inductive system such that greedy algorithm correctly computes \( \max_{F \in \mathcal{F}} \sum_{e \in F} w(e) \) for all \( w \), then \( M \) is a matroid.

**Pf:** Need to show that if \( F \in \mathcal{F} \), \( G \in \mathcal{F} \), and \( |G| > |F| \), \( \exists g \in G - F \) such that \( F \cup \{g\} \in \mathcal{F} \).

Let \( w(e) = \begin{cases} 1 & e \in F \\ 1 - \varepsilon & e \in G - F \\ 0 & e \notin G \cup F \end{cases} \)

The greedy algorithm first selects \( F \); but

\[ \sum_{e \in G} w(e) > |F| \text{ if } |G - F| (1 - \varepsilon) > |F - G| \]

\[ \therefore (1 - \varepsilon) > \frac{|F - G|}{|G - F|} \]

We can choose such an \( \varepsilon \) since \( \frac{|F - G|}{|G - F|} < 1 \)

\( \therefore \text{Alg. must continue to select more elements from } G - F \text{ after selecting } F \text{ and we are done.} \)
Edmonds' Proof Via LP Duality

1. Matroid Optimization

Ref. Matroids and the Greedy Algorithm
Jack Edmonds, Mathematical Programming,

Consider: \( x_j \geq 0 \) \( j \in E = \{1, 2, \ldots, n\} \)

\( \sum_{j \in E} x_j \leq r(s) \) \( \forall s \subseteq E. \)

\[ \text{Max} \sum_{j=1}^{n} w_j x_j \]

Since \( r(s) \in \{0, 1\}, 0 \leq x_j \leq 1 \) \( \forall j = 1 \ldots n \)

Integer solutions to \( P \) (necessarily 0/1 solution) is an indicator function of \( F \in \mathcal{F} \).

\[ y(s) \geq 0 \] \( \forall s \subseteq E \)

\[ \sum_{s \subseteq E} y(s) \geq w_j \] \( j = 1 \ldots n \)

\[ \text{Min} \sum_{s \subseteq E} r(s) y(s) \]

is the LP dual of \( P \).
Now let elements of $E$ be ordered so that

$$w_1 \geq w_2 \geq \ldots \geq w_m \geq 0 \geq w_{m+1} \geq \ldots \geq w_n.$$ 

Let $E' = \{1, 2, \ldots, m\} \subset E$.

Let $A_k = \{1, 2, \ldots, k\}$

Let $x^0 = (x_1^0, x_2^0, \ldots)$

where $x_i^0 = \gamma(A)\gamma(A)$, $x_j^0 = \gamma(A_j) - \gamma(A_{j-1})$, $j = 2, \ldots, m$

and $x_j^0 = 0$, $j > m$.

(This is the solution produced by the greedy algorithm)

It is easy to verify that $x^0$ is the indicator vector of a set $F$ which is a $M$-basis of $E'$.

Let $\{y^0(s)\}$ be defined as follows:

$$y^0(A) = \begin{cases} w_j & \text{if } j = j_1, \ldots, m-1 \\ w_m & \text{if } y^0(A_m) = \gamma(A_m) = w_m \\ 0 & \text{for all other } A \leq E. \end{cases}$$

Consider any element $j \in E$, $j \geq m$.

then $\sum_{j \in S} y^0(s) = 0 > w_j$
\[ \begin{align*}
\sum_{j \in A} y^0(j) &= \sum_{k=1}^{m} y^0(A_k) = \\
&= (\sum_{k=j}^{m} (W_k - w_{k+1})) + w_m \\
&= w_j \geq 0.
\end{align*} \]

Hence \( y^0 \) is feasible to the dual.

Check to see \((x^0, y^0)\) satisfy Complementary Slackness and hence optimal to \( P \) and \( \delta \) respectively.

[See page 279 of \textit{Cook, Cunningham, Pau and S}]