CS 6381 : Lecture # 28 11.2.2020

Many uses of Matroids :

1. Finding a directed spanning tree in a graph $G = [V; E]$ with edge wts $w(e)$, and root r.

   [An instance of 2-Matroid Intersection]

2. Given a Linear Program, check if it can be (legitimately) converted to a flow problem. If yes, find such a conversion. [Graphic Matroids]

3. Given a matrix, test if it is Totally unimodular [Regular Matroids]

4. Reconstruction of polyhedra from projected images (Sugihara)

5. Rigidity of Structures (Recski, Whitely)

6. Shannon Switching Game (Lehman, Edmonds)

7. Discrete Convexity

8. Matroid Parity Problem.

We will look at Matroid Intersection first.
Matroid Partition

See : [Cook, Cunninghan, P, S] and [Lawler], [J. Edmonds][21]

Polyhedral Characterization:

Let $[E, r] = M$ be a matroid.

Recall: $r : 2^E \to \mathbb{Z}_+$; $r(S) \leq |S| \ \forall S \subseteq E$,

r. is monotone nondecreasing as a set function

and is submodular.

Consider: $x_j \geqq 0$, $j \in E$

$$\left.\begin{array}{l} x_j \geqq 0, \quad j \in E \\ \sum_{j \in S} x_j \leqq r(S) \quad \forall S \subseteq E \end{array}\right\} \text{I)}$$

Any feasible integer solution satisfies $x_j \in \{0,1\} \ \forall j \in E$ (since $r(\{e\})$: 0/1)

Given 0/1 x^0 feasible to the above, let

$$F^0 = \{ j : x_j^0 = 1 \} \subseteq E.$$

$$r(F^0) \leq |F^0| = \sum_{j \in E} x_j^0 = \sum_{j \in F^0} x_j^0 \leq r(F^0)$$

since x^0 is feasible.

$\therefore$ equality holds throughout.

$\therefore F^0 \in \mathcal{I}$ where $\mathcal{I}$ are ind. sets of M.

System (I) written in matrix form.

$$x \geqq 0 \qquad x \in Z_+^E \text{ or } R_+^E$$

$$_{2^{|E|}}\overset{|E|}{A}\, x \leqq r$$

$\downarrow$

rows of A corresponds to subsets of E

elements are 0/1; indicating which elements belong to a particular subset of E; r is the rank function.

For any feasible x (not necessarily integer)

Consider the subset of constraints $\mathcal{S}$ that are satisfied as equations.

If $S \in \mathcal{S}, T \in \mathcal{S}$, then we have

$$\sum_{j \in S} x_j = r(S)$$

$$\sum_{j \in T} x_j = r(T)$$

$$\therefore \sum_{j \in S} x_j + \sum_{j \in T} x_j = \underbrace{\sum_{j \in S \cup T} x_j}_{\leq\, r(S \cup T)} + \underbrace{\sum_{j \in S \cap T} x_j}_{\leq\, r(S \cap T)} = r(S) + r(T)$$

$$\therefore r(S) + r(T) \leq r(S \cup T) + r(S \cap T)$$

But $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$

Since r is a rank function of a matroid and is therefore Submodular

$\therefore$ For such S and T,

$$r(S) + r(T) = r(S \cup T) + r(S \cap T)$$

$$\therefore \sum_{j \in S \cup T} x_j = r(S \cup T)$$

$$\sum_{j \in S \cap T} x_j = r(S \cap T)$$

$\therefore$ $S \cup T$, and $S \cap T$ also correspond to equality constraints and are in $\mathcal{S}$. $\therefore \mathcal{S}$ is a lattice under $\cup$ & $\cap$.

It is important to note that rows of A corresp. to $S, T, S \cup T$, ~~&~~ $S \cap T$ for any such family, are linearly dependent.

$$\boxed{A_{S.} + A_{T.} - A_{S \cup T} - A_{S \cap T} = 0}$$

Now about extreme points of polyhedron

$$\{x : x \geq 0, \sum_{j \in S} x_j \leq r(S) \ \forall S \subseteq E\}.$$

Lemma ~~A~~ (Linear Algebra / Programming)

Extreme points of $\{x : x \geq 0, Ax \leq r\}$ ~~are~~ correspond to <u>unique</u> solutions of the system

~~$\{x : x \geq 0, A_S x_S = r ; A_{\bar{S}} x_{\bar{S}} = 0 ; x_S \geq 0$~~

$$\{x : x_S > 0 ; x_{\bar{S}} = 0 ; A_T x = r_T ; A_{\bar{T}} x \leq r\}$$

S ~~is~~ $\subseteq E$; T a subset of rows of A,

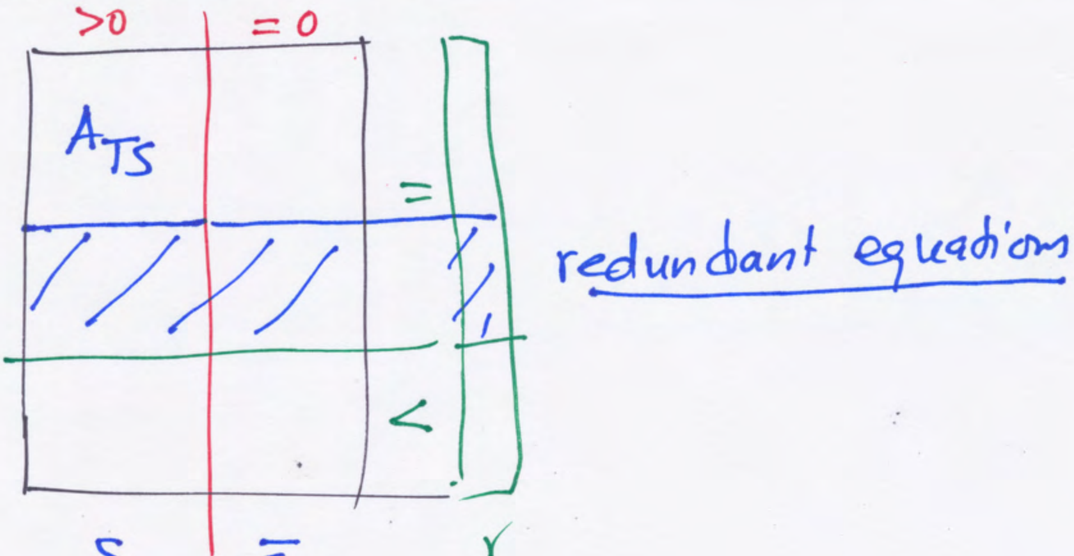
Such that A_{TS} is nonsingular.

S : Basis (LP sense) variables

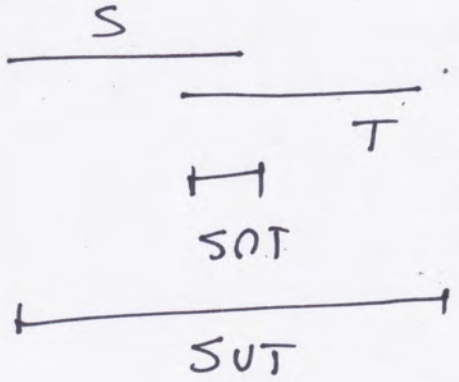
$\bar{S}$: Nonbasic Variables

A_T : Set of lin ind. rows among equality constraints for x.

A



In the matroid example, If $S, T, S\cup T, S\cap T$ are equalities, we need not keep ~~$S\cap T$~~ one of S,T; hence the remaining ones form a laminar sets



If we remove S or T we get <u>nesting</u>

$\therefore$ rows of A_T are either disjoint or nested. Such matrices are <u>totally Unimodular</u> (all subdeterminants are $0, +1, -1$)

Which gives integrality of extreme points

If $P_1 \subseteq R^n$ with integral extreme points &
$P_2 \subseteq R^n$ " "

Then it <u>may not be true</u> that $P_1 \cap P_2 \subseteq R^n$ has int. extreme points. Almost, the <u>only instance</u> where $P_1 \cap P_2$ has integer extreme points happens in Matroid Polyhedra. } Known as Matroid Intersection

Consider the system:

$$x_j \geqq 0 \quad \forall j \in E$$

$$\sum_{j \in S} x_j \leq r_1(S) \quad \forall S \subseteq E$$

$$\sum_{j \in S} x_j \leq r_2(S) \quad \forall S \subseteq E$$

Where $(E, r_1) = M_1$; $(E, r_2) = M_2$ are two matroids on the same set E. Let

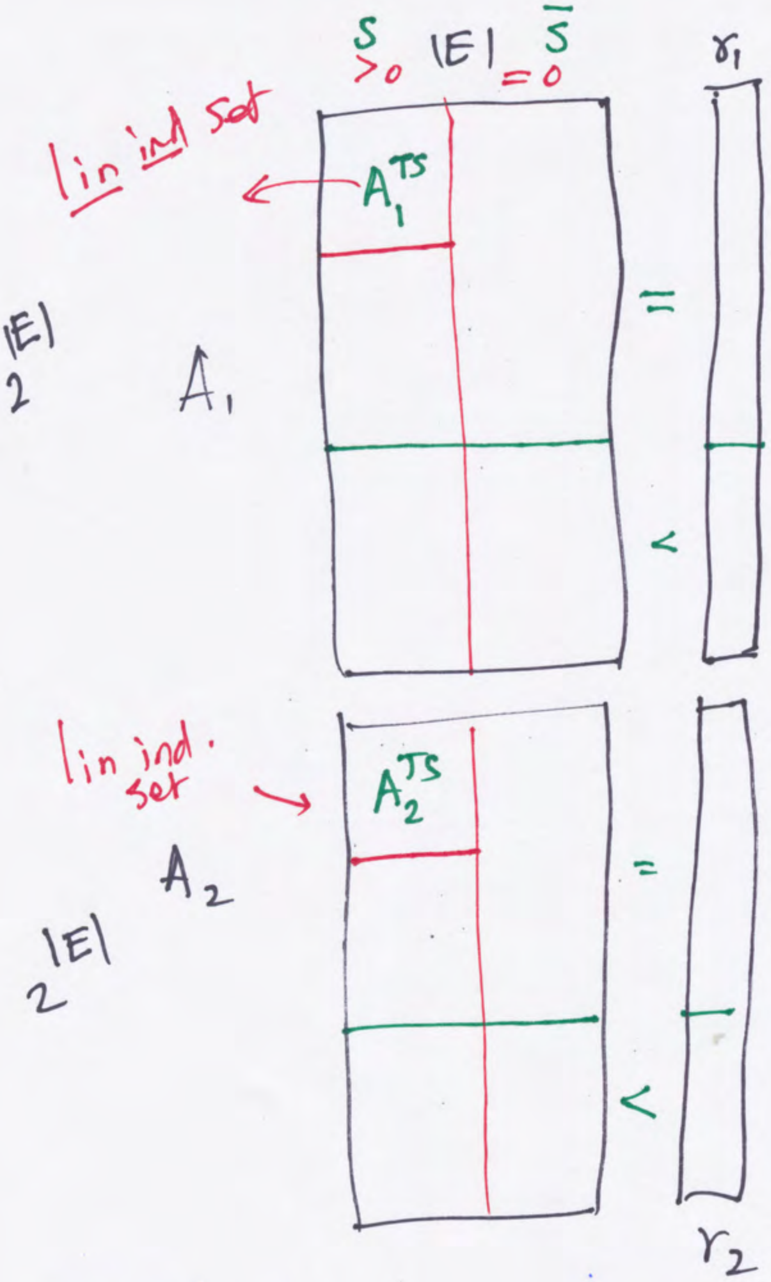
$$P_1 = \left\{ x \in R^{|E|} : x \geqq 0, \sum_{j \in S} x_j \leq r_1(S) \ \forall S \subseteq E \right\}$$

$$P_2 = \left\{ x \in R^{|E|} : x \geqq 0 \ \sum_{j \in S} x_j \leq r_2(S) \ \forall S \subseteq E \right\}$$

We Know from previous stuff, P_1 and P_2 have integer extreme points. J. Edmonds showed $P_1 \cap P_2$ has integer extreme points. There are two ways of doing this

1) Use submodularity and use laminarity to do this
2) Use a poly time Combinatorial algorithm to do it.

J. Edmonds did both.



Since A_1^{TS} is laminar & A_2^{TS} is laminar

↓

Can be converted to bipartite - transportation type set which is known to be <u>totally unimodular</u>

⇓

Ext-points of $P_1 \cap P_2$ are integral.

<u>Note</u> All we need is that r_1, r_2 are integer valued submodular functions that are non negative and monotone.

$\boxed{\text{Do Not need } r(S) \leq |S|}$

Such systems are called <u>poly-matroids</u>

Next we turn to some examples, and if time permits to Primal-Dual Alg. for this.