CS6381: Lecture # 29 11.9.2020

J. Edmonds starts with known results at that time (196x)

1. SP alg. for min (max) sp. tree (forest) in a weighted undir. graph. (Views as 1-Matroid optimization)
2. SP algorithm for ~~to~~ finding min(max) wted Perfect Matching in a bipartite graph. (Which he views as a special case of two problems: (a) Matroid Intersection ~~(b)~~ (b) Min (max) wted PM in a general graph)

We have already seen 2(b).

3. SP alg. for ~~min (max)~~ Optimal Branchings directed maximal ~~s~~ forest (sp. tree) in a weighted dir. graph (This he views as a special case of Matroid Intersection Problem)

4. Independence "Oracle" for Matroid Union using Oracles for each matroid: Matroid PARTITION – closely related to Matroid Intersection via Matroid Dual.

Matroid Intersection (Algorithms)

Let $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ be two matroids on the same set E.

Problem (Cardinality version):

Find $F \in \mathcal{I}_1 \cap \mathcal{I}_2$, with max $|F|$.

Preliminary results:

Let $F \in \mathcal{I}_1 \cap \mathcal{I}_2$, $S \subseteq E$.

Since $\mathcal{I}_1, \mathcal{I}_2$ give rise to ind. systems on E,

$$F \cap S \in \mathcal{I}_1$$

$$F \cap \underline{E - S = \bar{S}} \in \mathcal{I}_2$$

$$\therefore |F| = |F \cap S| + |F \cap \bar{S}| \leq r_1(S) + r_2(\bar{S}) \quad (*)$$

$\therefore$ If we can find $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ and S such that equality holds in $(*)$, $|F|$ is maximum.

Thm: (J. Edmonds)

$$\max\{|F| : F \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{S \subseteq E}\{r_1(S) + r_2(\bar{S})\}$$

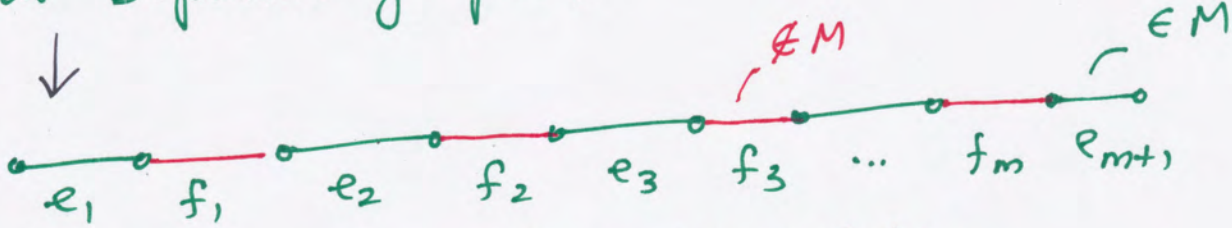
Proof via an algorithm.

(There is also more than one non-alg. proof)

Cardinality Matroid Intersection Algorithm

Uses an "augmenting path" approach and hence is a "primal" algorithm.

(The algorithm is like Cardinality matching in bipartite graphs)



(aug. path in bipartite matching

- $e_i \notin M$
- $f_i \in M$
- ~~Mo~~ + additional properties. (See CCPS page 289)

Given $M_1 = [E, \mathcal{I}_1]$, $M_2 = [E, \mathcal{I}_2]$ and $F \in \mathcal{I}_1 \cap \mathcal{I}_2$.

We try to augment F. For this, we construct a dir graph $G(M_1, M_2, F) = [V; U]$

$V: \{E \cup \{r, s\}\}$; U:

M1-edges • (e, s) : for every $e \in \bar{F}$ such that $F \cup \{e\} \in \mathcal{I}_1$

M2-edge • (r, e) : for every $e \in \bar{F}$ " $F \cup \{e\} \in \mathcal{I}_2$

M1-edges • (e, f) : $e \in \bar{F}$, $f \in F$: $F \cup \{e\} \notin \mathcal{I}_1$, $F \cup \{e\} - \{f\} \in \mathcal{I}_1$

M2-edges • (f, e) : $e \in \bar{F}$, $f \in F$; $F \cup \{e\} \notin \mathcal{I}_2$, $F \cup \{e\} - \{f\} \in \mathcal{I}_2$

Note: If $(e,s) \in U$, $\nexists$ any other edge of the form (e, x) and if $(r,e) \in U$, then $\nexists$ any edge of the form (x, e).

1. form (e, x) and if $(r,e) \in U$, then $\nexists$ any edge of the form $(\cancel{\;}x,e)$.

2. If $(e, f) \in U$, $f \in F$, $e \notin F$, then $(e, s) \notin U$ and if $(f, e) \in U$, $e \notin F$, then $(r, e) \notin U$.

3. The dir. edges of this graph are of two types: those determined by M_1 and those determined by M_2. These are called M_1-edges and M_2-edges.

4. Construction is NOT Symmetric w.r.t to M_1 and M_2. (but reversing directions and the role of r and s, has the effect of exchanging M_1 and M_2.)

A different form of symmetry will be used later.

Examples on next page (page 290 CCPS)

One where $|F|$ is max and one where $|F|$ is not maximum.

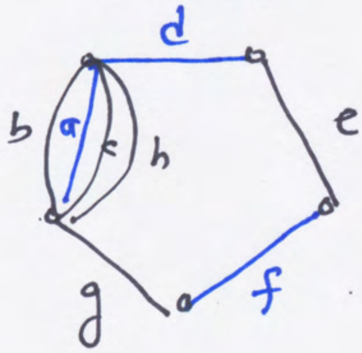
In these examples, M_1, M_2 are <u>forest</u> matroids on G_1, G_2 respectively.

~~E_F~~ F edges are shown in blue.

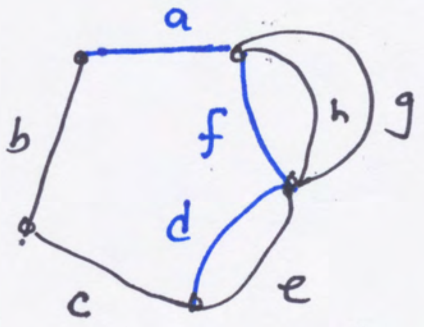
<u>Example I</u>

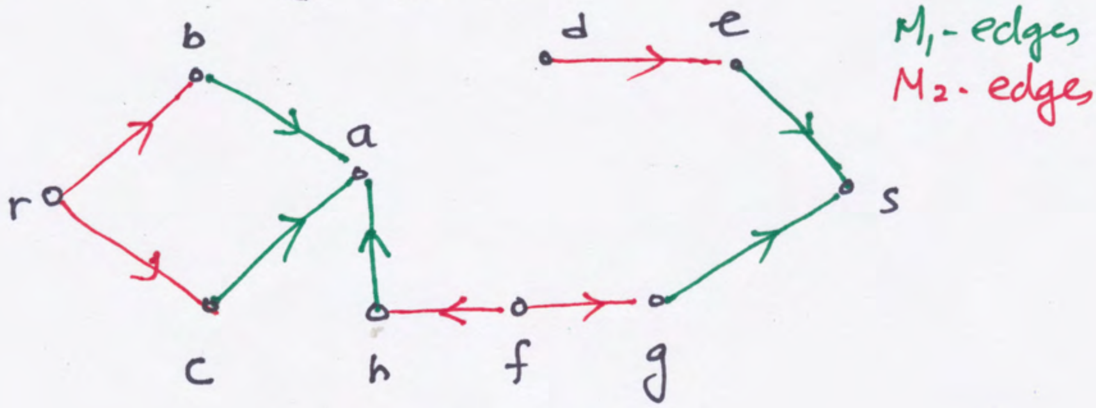
$F = \{a, d, f\}$

G_1



G_2





$G(M_1, M_2, F)$

<u>Note</u>: There is no dir. path from r to s in $G(M_1, M_2, F)$ in this example.

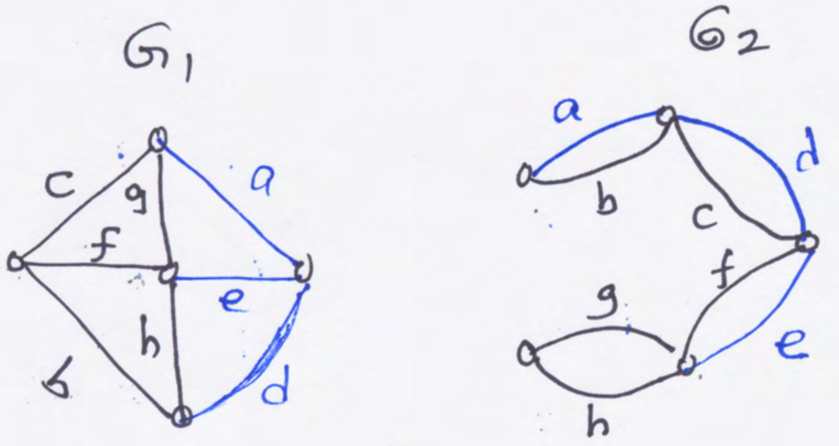
Let $A = \{b, c, a\}$;

<u>A is the set of nodes in $G(M_1, M_2, F)$ reachable from r.</u>

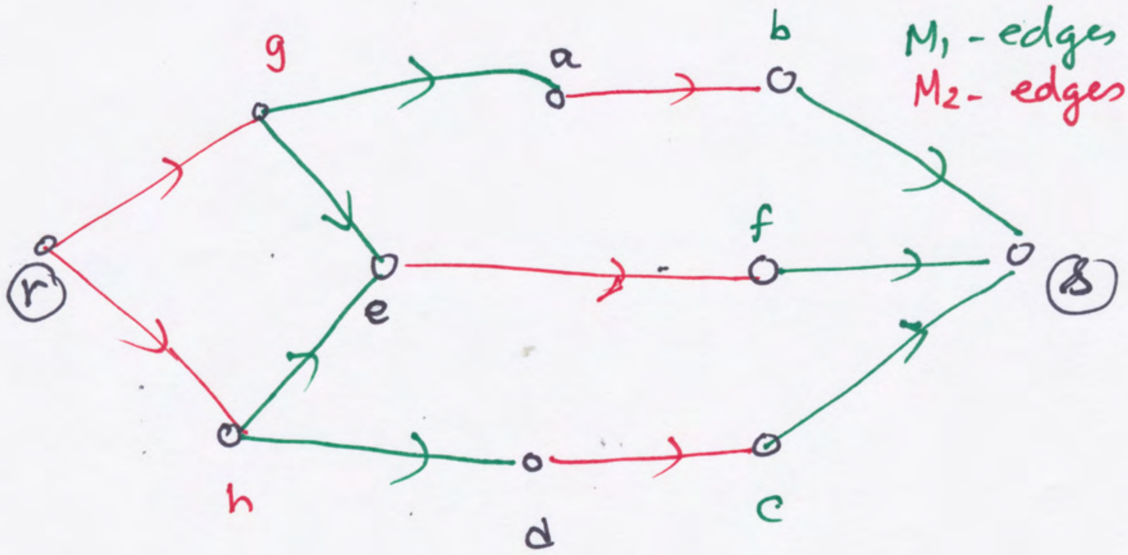
We can show $|F| = r_1(A) + r_2(\bar{A})$

Showing $|F|$ is maximum.

Example II



$F = \{a, d, e\}$



$G(M_1, M_2, F)$

important

Chordless r–s dir. path.

r g a b s is such a path

$F = \{a, d, e\} \in \mathcal{F}_1 \cap \mathcal{F}_2$

$\hat{F} = \{a, d, e\} + \{g\} - \{a\} + \{b\}$

$= \{b, d, e, g\} \in \mathcal{F}_1 \cap \mathcal{F}_2$

$|\hat{F}| = |F| + 1$.

Theorem: (in two parts)

(a) If there is no (r, s) path in $G(M_1, M_2, F)$ then $|F|$ is maximum; in fact, if $A \subseteq E$, $\exists$ no out going edge from $A \cup \{r\}$ in $G(M_1, M_2, F)$ then, $|F| = r_1(A) + r_2(\bar{A})$

Proof: [see 291 in CCPS]

Let $e \in A - F$. Since (e, s) is not present in $G(M_1, M_2, F)$, $F \cup \{e\}$ contains a M_1circuit C.

Since $\exists$ no edge (e, f) in $G(M_1, M_2, F)$ with $f \in \bar{A}$, we have $C \subseteq (A \cap F) \cup \{e\}$

$\therefore$ $F \cap A$ is an M_1-basis of A.

Similarly $F \cap \bar{A}$ is an M_2- basis of $\bar{A}$

Hence $|F| = |F \cap A| + |F \cap \bar{A}|$

$$= r_1(A) + r_2(\bar{A})$$

Hence $|F|$ is max size member of $\mathcal{F}_1 \cap \mathcal{F}_2$.

Part (b) of theorem refers to increasing size on the next page.

(b) If there is an $(r\text{-}s)$-dir. path in $G(M_1, M_2, F)$ without chords (fewest edges), let such a path look like

$$r \to e_1 \to f_1 \to e_2 \to f_2 \to \cdots \to e_m \to f_m \to e_{m+1} \to s$$

<u>Def</u>: <u>Span</u> of a set $A \subseteq E$ in a matroid $M = [E, \mathcal{J}]$ is given by

$$Sp(A) = \left\{ e : e \in A \text{ or } e \notin A,\ e \in C \subseteq A \cup \{e\} \text{ where } C \text{ is a circuit of } M \right\}$$

<u>Lemma</u>: Let $F \in \mathcal{J}$, for a matroid $M = [E, \mathcal{J}]$.

Let $p_1, q_1, p_2, q_2, \ldots p_k, q_k$ be a sequence of <u>distinct</u> elements of E such that

(i) $p_i \notin F$, $q_i \in F$ for $1 \le i \le k$

(ii) $F \cup \{p_i\} \notin \mathcal{J}$; $F \cup \{p_i\} - \{q_i\} \in \mathcal{J}$, $1 \le i \le k$

Chordless $\leftarrow$ (iii) $F \cup \{p_i\} - \{q_j\} \notin \mathcal{J}$ $1 \le i < j \le k$.

Let $F' = F \,\Delta\, \{p_1, q_1, \ldots p_k, q_k\}$.

Then $F' \in \mathcal{J}$ and $sp(F') = sp(F)$.

Relevance of the Lemma.

With respect to Matroid M_1, $F \in \mathcal{I}_1 \cap \mathcal{I}_2$, and the sequence $e_1, f_1, e_2, f_2, \ldots e_m, f_m$ in the above (r-s) dir. path in $G(M_1, M_2, F)$, Conditions of the lemma are satisfied.

Since there is no chord (e_i, s), (ii) is satisfied

Since " " " $(e_i, f_j), i<j$, (iii) is satisfied.

Hence from the lemma, we can conclude that

$F' = F \cup \{e_1, e_2, \ldots e_m\} - \{f_1, \ldots f_m\} \in \mathcal{I}_1$

and $Sp_1(F') = Sp_1(F)$ (Refers to span w.r.t M_1)

Since $e_{m+1} \notin Span_1(F)$, $e_{m+1} \notin span_1(F')$

Hence, $F' \cup \{e_{m+1}\} \in \mathcal{I}_1$

Watch out. Going backwards (backward symmetry with edge dir. reversed) yields this set $F' \cup \{e_{m+1}\} \in \mathcal{I}_2$.

Now we turn to proof of lemma.

Proof of Lemma:

By induction on k;

$k=1$: $F' \in \mathcal{J}$ by (ii) of Lemma.

Suppose $g \in Sp(F)$, $g \notin Sp(F')$. Then F is M-basis of $F' \cup \{p_1, g\}$ but $F' \cup \{g\}$ is another M-basis of larger size — a contradiction.

Suppose $g \in Sp(F')$, $g \notin Sp(F)$ then F' is an M-basis of $F' \cup \{q_1, g\}$ but $F \cup \{g\}$ is a also M-basis of this set but has a larger size — a contradiction.

Induction Step: Let $k \geqq 2$ and the result is true for smaller values. Consider F and $\{p_1, q_1, \dots p_{k-1}, q_{k-1}\}$ use I.H as well as $F - \{q_k\}$ and $\{p_1, q_1, \dots p_{k-1}, q_{k-1}\}$.

Hence, (i) $F' \cup \{q_k\} - \{p_k\} \in \mathcal{J}$, $Sp(F' \cup \{q_k\} - p_k) = Sp(F)$

(ii) $F' - \{p_k\} \in \mathcal{J}$, $Sp\{F' - \{p_k\}\} = Sp\{F - \{q_k\}\}$

Recall: $F' = F \,\Delta\, \{p_1, q_1, \dots \cancel{p_k, q_k}\ p_{k-1}, q_{k-1}\}$

~~Since~~ $p_k \notin Sp(F - \{q_k\})$; hence $p_k \notin Sp(F' - \{p_k\})$ (ii)

and hence $F' - \{p_k\} + \{p_k\} = F' \in \mathcal{J}$

$Span(F') = Sp\{F' \cup \{q_k\} - \{p_k\}\}$

from Case 1 applied to J', q_k, p_k;

k-1 (from^{IH} $J' \cup \{q_k\} \notin \mathcal{J}$, $J' \cup \{q_k\} - \{p_k\} \in \mathcal{J}$

Case and (ii)

I am not happy with this ; see CCPS 291-293

$\therefore F \in \mathcal{F}$.

I.H: $F^j = F \Delta \{p_1, q_1, \cdots p_j, q_j\} \in \mathcal{F}, 1 \le j \le k-1$

(i) $F^{k-1} = F^k \cup \{q_k\} - \{p_k\} \in \mathcal{F}$ by I.H.

$\& \; Sp(F^{k-1}) = Sp(F)$

(ii) $\hat{F}^j = F^j - \{p_j\}$; $\cancel{\hat{F}^k = F^k - \{p_k\}}$ $=$

$\cancel{E \& \hat{F}^k - \{p_k\} \in \mathcal{F}}$,

$\cancel{= F^{k-1}}$

Consider $F - \{q_k\}$ & sequence

$\{p_1, q_1, \cdots p_{k-1}, q_{k-1}\}$

$(F - \{q_k\}) \; \Delta \; (p_1, q_1, \cdots p_{k-1}, q_{k-1})$

$= F^{k-1} - \{q_k\} \in \mathcal{F}$ by I.H.

$= F^{k} - \{p_k\}$

$\& \; Sp(F^k - \{p_k\}) = Sp(F - q_k)$ by previous results & I.H.

Since $p_k \notin Sp(F - q_k)$ and hence $p_k \notin Sp(F^k - \{p_k\})$

So $F^k - \{p_k\} + \{p_k\} = F^k \in \mathcal{J}$.

To prove $Sp(F^k) = Sp(F)$, it suffices to show

$$Sp(F^k) = Sp(F^k + \{q_k\} - \{p_k\})$$

For this, apply $k=1$ to $\cancel{F^{k-1}}$,

$F^k, \; q_k, \; p_k$

Recall $\left.\begin{array}{l} F^k \cup \{q_k\} \notin \mathcal{J} \\ F^k \cup \{q_k\} - \{p_k\} \in \mathcal{J} \end{array}\right)$ from (i).

Still not happy ; CCPS use $\mathcal{J}'$ for F^k; but seems to be confusing.

Weighted Matroid Intersection

LP: $x \in \mathbb{R}_+^E \; (x \geq 0)$,

$y_1(A) \geq 0$: $x(A) \leq r_1(A) \quad \forall A \subseteq E \; (M_1)$

$y_2(A) \geq 0$: $x(A) \leq r_2(A) \quad \forall A \subseteq E \; (M_2)$

$$\text{Max} \sum_{e \in E} \cancel{w(e)}^{\,c(e)} x_e \qquad (P)$$

(We may assume without loss, $w(e) > 0 \;\forall e \in E$)

$$y_1(A) \geq 0,\; y_2(A) \geq 0 \quad \forall A \subseteq E,$$

$$\sum_{A : e \in A} \left(y_1(A) + y_2(A) \right) \geq \cancel{w(e)}\; c(e) \quad \forall e \in E$$

$$\text{Min} \sum_{A \subseteq E} \left[r_1(A) y_1(A) + r_2(A) y_2(A) \right]$$

Weight splitting Idea.

Let y_1^*, y_2^* be an optimal solution to D

and let x^* be " " .. P

($\downarrow$ indicater vector for $F^* \in \mathcal{F}_1 \cap \mathcal{F}_2$)

Define $c_1, c_2 \in \mathbb{R}^E$ as follows

$$c_1(e) = \sum_{A : e \in A} y_1^*(A); \quad c_2 = c - c_1$$

<u>Lemma</u>: Suppose y_1^*, y_2^* is optimal to Ⓓ (of the Matroid intersection problem). Then y_i^* is optimal to Ⓓ of M_i - optimization problem. Conversely, if y_i^* is optimal for M_i - optimization's dual for $i=1,2$, Then (y_1^*, y_2^*) is optimal to dual of matroid intersection.

Let $c_1 + c_2 = c$, be a weight splitting.
Suppose a weight splitting and $F^* \in \mathcal{F}_1 \cap \mathcal{F}_2$ one such that F^* is optimal to 1-Matroid optimization for c_i in Matroid M_i, $i=1,2$
Let $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ be another common indep. set.

$$c(F^*) = c^1(F^*) + c^2(F^*) \geq c^1(F) + c^2(F) = c(F)$$

$\therefore F^*$ is optimal for weighted intersection problem.
(This is useful in <u>Certificate</u> to show optimality)

<u>Result</u>: $\exists$ a $F^* \in \mathcal{I}_1 \cap \mathcal{I}_2$ and a weight splitting c^1, c^2 such that F^* is c^i-optimal for M_i $i = 1, 2$.

<u>Algorithm I</u> (Weighted Matroid Intersection)

Extension of Cardinality case.

<u>Input</u> ~~M_i, c^i~~ Matroids M_i $i = 1, 2$,

$F \in \mathcal{I}_1 \cap \mathcal{I}_2$,

$G = (M_1, M_2, c, F)$ is defined as follows

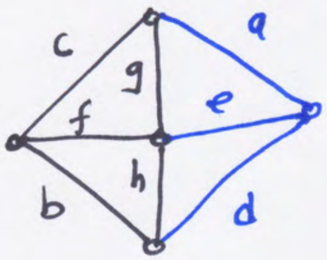
$G = (M_1, M_2, F)$ as before.

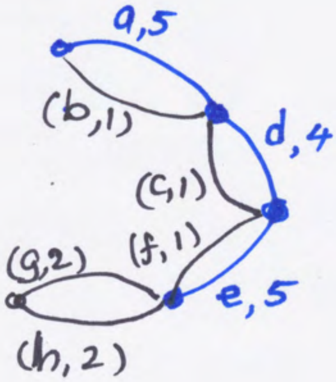
Costs of edge (edge weights): $p_{v,w}$:

- $p_{e,f} = c_f - c_e$ for each M_1-edge (e,f) with $e \notin F$, $f \in F$
- $p_{e,s} = -c_e$ for each M_1-edge (e,s) $e \notin F$
- $p_{v,w} = 0$ for each M_2-edge (v,w).

An example on the next page

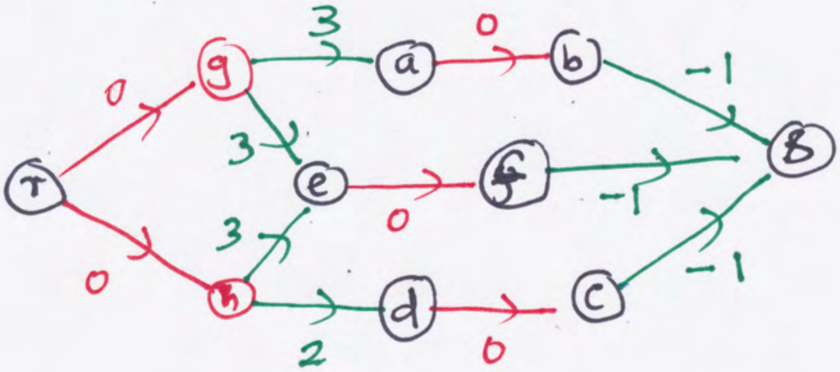
$G_1 \to M_1$ $\quad$ $G_2 \to M_2$





$F = \{a, d, e\}$

max wt common indep. set of size 3



Assuming F is a maximum weight common independent set of size $|F|$, we find (if it exists) a least weight $(r-s)$ path (and if there are ties, select a path with as few edges → will result in a "chordless" path)

In the above example:

$$r \to h \xrightarrow{2} d \to c \xrightarrow{-1} s \quad : \text{total: } 1$$

$F' = F \Delta \{h, d, c\} = \{a, e, h, c\}$

↓

min wt common ind. set of size $|F|+1$.

If at some size, total weight goes down, previous one is the answer to the overall problem.

In our example, $\{a, d, e\}$ has max wt among common indep. sets.

The above does not provide dual solution or Certification as is done in Primal-Dual algorithms. This is done via weight splits as additional outputs.

At any stage, let $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ and (c_1, c_2) be a weight splitting. Let $C_0^i = \max\{c_i^e : e \notin F, F \cup \{e\} \in \mathcal{I}_i\}$ $i = 1, 2$.

[If these are not well defined, F is of max. cardinality]; we do not need to look for a larger set of maximum weight and algorithm stops].

$G(M_1, M_2, F, c^1, c^2)$ are defined in the next page.

$G = G(M_1, M_2, F)$ as before.

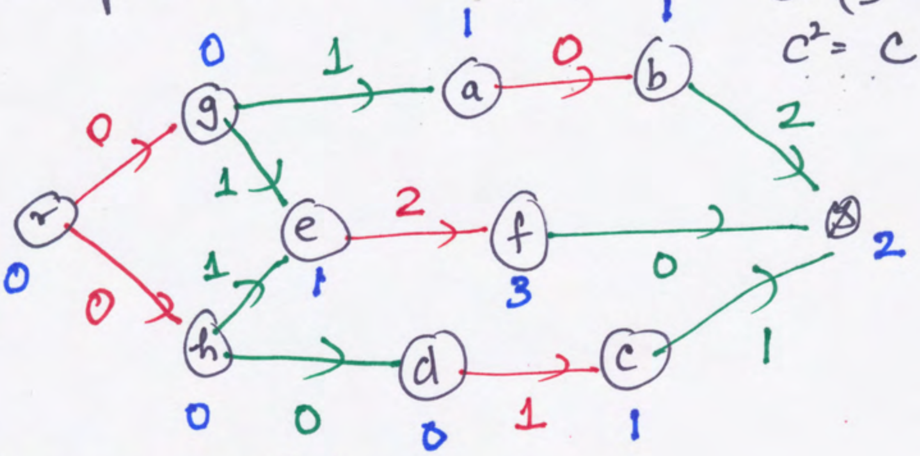
Costs (weights) $p_{u,v}$ are defined by:

- $p_{e,s} = C_0^1 - C_e^1$ for each M_1-edge (e,s) with $e \notin F$
- $p_{r,e} = C_0^2 - C_e^2$ for each M_2-edge (r,e) with $e \notin F$
- $p_{e,f} = -C_e^1 + C_f^1$ for each M_1-(e,f) edge ; $e \notin F$ $f \in F$
- $p_{f,e} = -C_e^2 + C_f^2$ for each M_2-edge ; $e \notin F$ $f \in F$

For the previous example

$$\begin{array}{cccccccccc} & a & b & c & d & e & f & g & h \\ C^1 = (& 3 & -1 & 0 & 2 & 3 & 1 & 2 & 2) \end{array}$$

$C^2 = C - C^1$



d[v] shown in blue

Please see CCPS p. 304 on