Now we turn to Q1: Send flows so as to maximize surviving flow.

For this, we have to look into how an adversary would think. Given a feasible flow, which edge(s) should the adversary destroy, if their number is restricted to k.

We consider the case $k=1$.

Given a feasible flow $\{f, F\}$, if there are cycles (directed) carrying positive flow, we can reduce flows on edges of cycle without affecting F until at least one of the edges in such a cycle has 0 flow.

So from now on we will assume, that the flow selected is such that $\nexists$ cycles of positive flow i.e flow is acyclic. In this case any decomposition will use up all the flow on edges.

If we now destroy an edge, the "loss" will equal flow on this edge. With this, adversary will destroy ~~the~~ an edge with maximum flow in order to decrease F by the largest amount.

Now we are ready for our formulation on how to choose f to maximize surviving flow. This in turn leads us to Parametrized flows Problem described below

Parametrized Max-Flow Problem:

$$\boxed{P(\lambda)} \quad \begin{cases} 0 \le f_{i,j} \le u_{ij} \quad \forall (i,j) \in E \\ \qquad f_{ij} \le \lambda \quad \forall (i,j) \in E) \end{cases}$$

$$\sum_j f_{i,j} - \sum_j f_{j,i} = \begin{cases} F(\lambda) & i = s \\ 0 & i \ne s, t \\ -F(\lambda) & i = t \end{cases}$$

$$\text{Maximize} \left[F(\lambda) - \lambda \right]$$

Consider $P(\lambda)$:

Same as above except

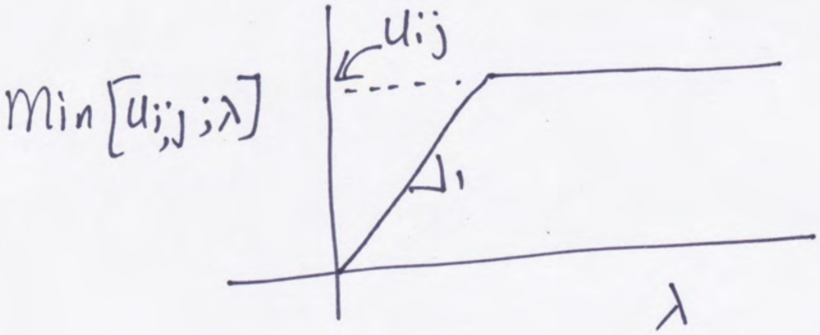
Maximize $F(\lambda)$

$$0 \le f_{i,j} \le \text{Min}\left[u_{ij} ; \lambda \right] \quad \forall (i,j) \in E$$

We first solve this & use it to answer Q1, and Q2.

$P(\lambda)$

Some properties:



$\forall (i,j) \in E$.

Concave, piecewise linear

$F(\lambda) \uparrow$ as $\lambda \uparrow$ (weakly)

Non-decreasing

$F(\infty)$: max flow with u_{ij} $\forall (i,j) \in E$.

$F(\varepsilon)$: max flow with $u_{ij} = \varepsilon$ $\forall (i,j) \in E$

for small enough ϵ.

Both of these are easily computed.

Capacity of any cut (separating s from t) is a sum of capacities of edges in the cut (going from s side to t side)

Sum of piecewise linear functions is piecewise linear; Sum of concave functions is Concave.

$\therefore$ Capacity of each such cut is Concave, piecewise linear

Minimum of Concave functions that are piecewise linear, is also piecewise linear and Concave.

$\therefore$ Min cut capacity is a piecewise linear Concave function of λ

But Max Flow = Min cut.

$\therefore$ $F^*(\lambda)$ is piecewise linear and Concave.

For Such functions, slopes are non-increasing. These slopes correspond to # of edges in min cut that have $\min[u_{ij}, \lambda] = \lambda$. Hence, Slopes are integers bounded by a polynomial in # of edges. We can get a better bound. When we solve for $F^*(\varepsilon)$, $\varepsilon > 0$ small, since all Capacities are equal to ε, (the problem is equi-valent to having all capacities equal to 1), the ~~slope of~~ max value $F(\lambda)$ for λ very small Corresponds to number of edges in a min cut multiplied by λ. The Min cut has no more edges than the cut with s alone on one side and this is $\leq |V|-1$.

Since $F^*(\lambda)$ is Concave, ~~pie~~ this is the largest slope. Hence each slope of the piecewise linear Concave function $F^*(\lambda)$ is an integer $\le |V|-1$. Hence, there are no more than $|V|-1$ distinct pieces of this function. Now to the process of determining $F^*(\lambda)$ as a function of λ.

We will explain this with an example:

(See next page) for this.

The use of $F^*(\lambda)$ curve for solving Q2.

Recall Kishimoto's formulation:

$$0 \le f_{i,j} \le u_{i,j} \quad \forall (i,j) \in E$$

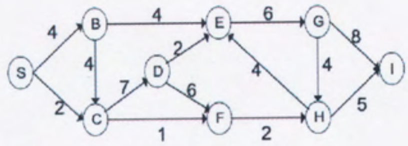
$$\sum_j f_{i,j} - \sum_j f_{j,i} = \begin{cases} F & i = s \\ 0 & i \ne s, t \\ -F & i = t \end{cases}$$

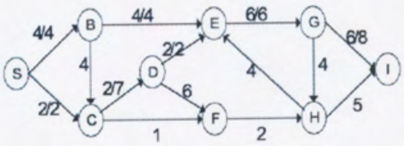
$$f_{i,j} \le \frac{F}{q} \quad \forall (i,j) \in E$$

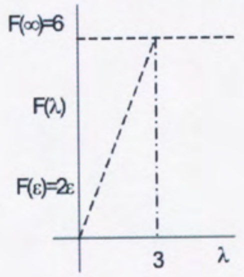
$$\text{Max } F \text{ or Max } \frac{F}{q}.$$

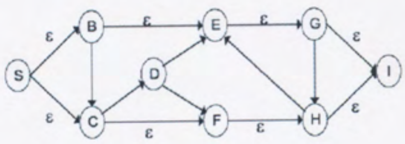
Think of $\frac{F}{q}$ as λ in $F(\lambda)$ problem.

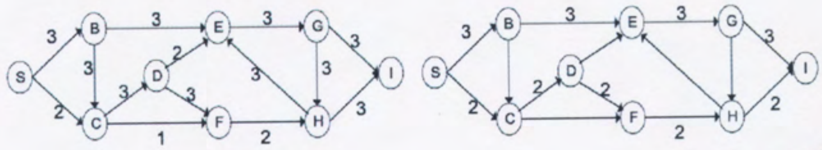
Example for $F^*(\lambda)$, $\max_{\lambda}\{F(\lambda)-\lambda\}$

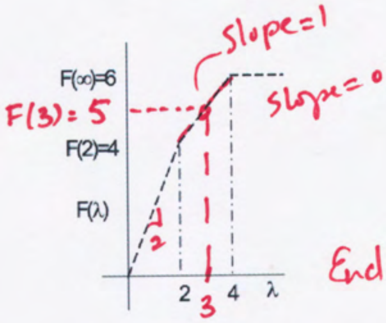


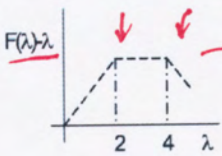






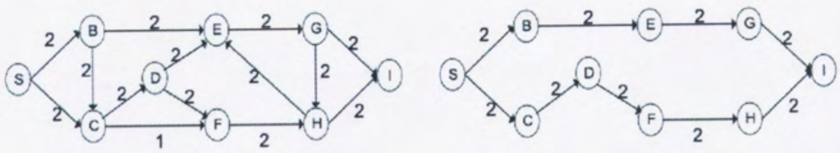






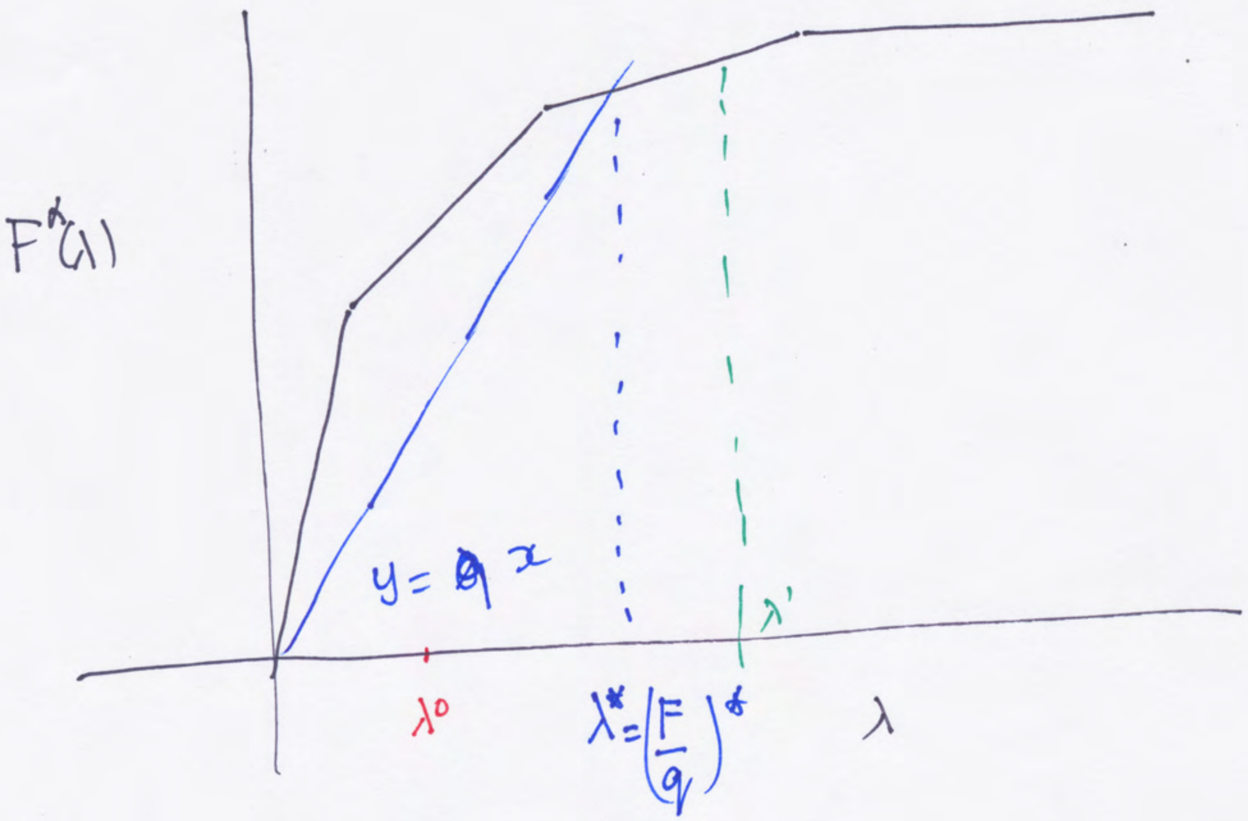
optimal λ & $F(\lambda)$ which max $F(\lambda)-\lambda$

max both $F^*(\lambda)$ and $F(\lambda)-\lambda$ simultaneously



$\lambda^* = 2$ is used here.

$\lambda^* = 4$ will give 4 units on upper path but if adv. destroys one edge in that path, $F(\lambda)-\lambda = 2$ again



$F(\lambda^0) \geqq q\lambda^0 \quad \therefore \quad \lambda^0 \leq \frac{F(\lambda^0)}{q}$

$\therefore \; f_{ij}(\lambda^0) \leq \frac{F(\lambda^0)}{q} \quad \forall (i,j) \in E$

$F(\lambda') < q\lambda' ; \; \therefore \; \lambda' > \frac{F(\lambda')}{q}$

and if $f_{ij}(\lambda') = \lambda'$ this violates feasibility

Max λ such solution is feasible is λ^*.

Once we get λ^*, and $f_{ij}(\lambda^*)$ this can now be decomposed into q-path flows as before using Matrix-decomposition based on Birkhoff's Thm