Proof of Correctness of Multiterminal Flow Algorithm

Let us look at the process in general at general step.

Suppose the structure looks like diagram below at some step.

Each $X_i$ and $y$ are sets of nodes.

$V_j$: max flows in previous steps (and min-cuts)

Suppose now we select a pair of nodes in $Y$ for the next step.

When $Y$ is removed from the above diagram we get 3 connected pieces: $X_2 U X_3$, $X_4 U X_5 U X_6$

Each of these is (separately) condensed to create the new condensed network. $Y$ is not condensed.
Suppose this splits \( Y \) into \( Y_1 \) and \( Y_2 \).

New max flow \( \leftarrow V_4 \)

= min cut

\((x_4, x_5, x_6)\) is on the side of \( Y_1 \)

\((x_1)\), and \((x_2, x_3)\) are on the side of \( Y_2 \).

This process is repeated until all sets are singleton node sets. At this time we have the spanning tree that is "equivalent" to the original network. Now \( \nu_{ij} = \min \nu_{k, l} \) \( \forall i, j \).

Proof of this follows.
Consider two nodes \( x \) and \( y \). In the final tree, removal of any edge partitions the nodes with \( x \) and \( y \) on opposite sides. Moreover, the value of the edge equals the cut corresponding to this partition. This cut, therefore, separates \( x \) from \( y \). \( \therefore \delta_{x,y} \leq \text{value of this cut} \).

If this edge represented by this cut is on the unique path in the tree from \( x \) to \( y \).

\( \therefore \cup_{x,y} \leq \cup_{(k,l) \in P \cap E} \forall i,j \)

\[ (1) \quad \cup_{x,y} \leq \min \cup_{k,l} \delta_{k,l} \quad \forall i,j \text{ } \]

Now we show the opposite inequality. For this purpose we need a lemma.

**Lemma**: At any stage of the construction, if there is an edge connecting sets \( X \) and \( Y \) in the tree, \( \exists \ x \in X, \ y \in Y \) such that \( \cup(x, y) = \delta \)

And the cut obtained by removing this edge is a min-cut in the original graph between \( x \) and \( y \).
Proof: By induction.

**Basis Case:** At the first step, this is "clearly" true by construction.

Suppose by way of Induction Hypothesis, it is true up to some point. We now select two nodes from a set $S$; say $Y$ as in previous diagrams.

![Diagram](before)

Let $v = v(x,y)$

Let $s,t \in Y$ be the next pair selected.

Suppose $Y$ splits into $Y_1, Y_2$ with $x$ on the side of $Y_1$.

![Diagram](after)

$v' = v(s,t)$ by construction.

**Case 1:** $y \in Y_1$, easy; $v = v(x,y)$ as before.

**Case 2:** $y \in Y_2$ (more difficult case).
Claim: \( v = v(s, x) \) in this case.

\[ \text{Pf: } \] \( x \) and \( s \) are on the same side of \( st \) mincut that gave rise to \( u' \).

So, when solving the \( s + s, x \) problem, condensing \( y_2 \) does not change max-flow, by Lemma shown before.

Let \( \overline{v}(i,j) \) : represent max flow in original network

\( \overline{v}(i,j) \) : represent max flow in \( Y_2 \) condensed network.

We have

\[
\overline{v}(x, s) = v(x, s) \\
\overline{v}(x, y) \geq v(x, y) = v \\
\overline{v}(y, t) = \infty \\
\overline{v}(t, s) \geq v(t, s) = v' 
\]

\[
v(x, s) = \overline{v}(x, s) \geq \min \left[ \overline{v}(x, y), \overline{v}(y, t), \overline{v}(t, s) \right] \\
= \min \left[ \overline{v}(x, y), \overline{v}(t, s) \right] \\
\]

\[
\overline{v} \quad \overline{v}'
\]
But the cut with capacity \( u' \) (in this case)
also separates \( x \), and \( y \).
\[
\therefore u' \geq u(x,y) = v
\]
\[
\therefore \text{Min } [u', v] = v.
\]
\[
\therefore u(x, y) \geq v
\]
But this cut with capacity \( v \) also separates \( x \) and \( y \) in this Case.
\[
\therefore u(x, y) \leq v
\]
\[
\therefore u(x, y) = v \text{ as claimed.}
\]

This completes proving claim.

\[
\therefore \text{Each edge in the final graph represents Max Flow (and Min-cut) for the node that form this edge.}\]
\[
\therefore \text{Flow values are correct for each pair connected by an edge in the final tree.}
\]
\[
\therefore u(i,j) \geq \text{Min } [u(i,k), u(k,l), \ldots u(x,j)]
\]
for \( i, j \)
(I) and (II) give the result stated at the beginning.

Hence, the number maximum flow problem done is \( |V| - 1 \), each some of which are on small graphs (because of condensing).

Condensing is essential to preserve the non-crossing nature of the cuts.

This tree is not only "flow equivalent" but also cut preserving. And hence is called a cut-tree equivalent to the original problem.

This completes the analysis problem.

Now we turn to the Design or Synthesis problem in the next section.