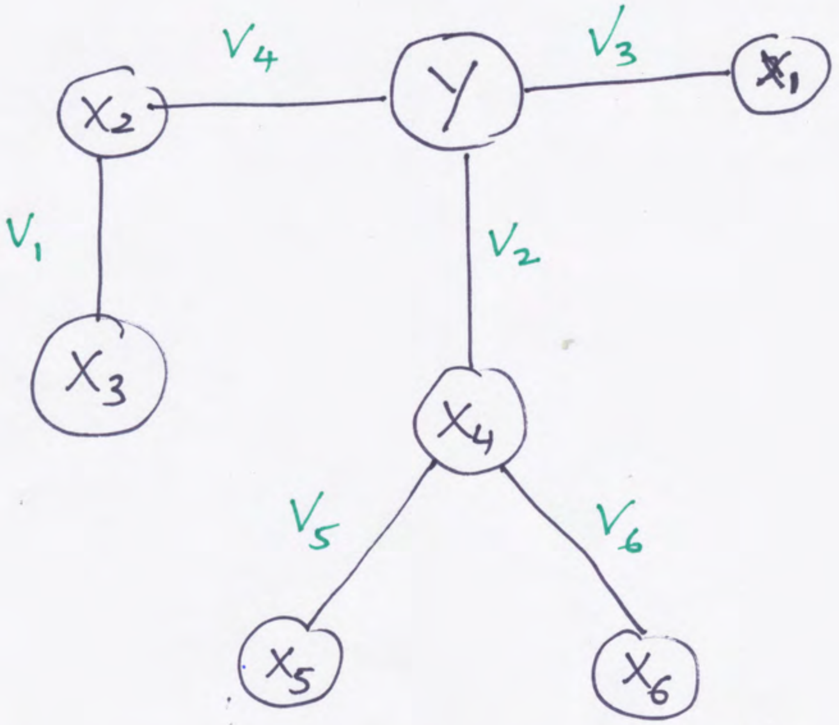
Proof of Correctness of Multiterminal Flows Algorithm

Let us look at the process in general at general step.

Suppose the structure looks like diagram below at some step

Each X_i and Y are sets of nodes

V_j : max flows in previous steps (and min-cuts)

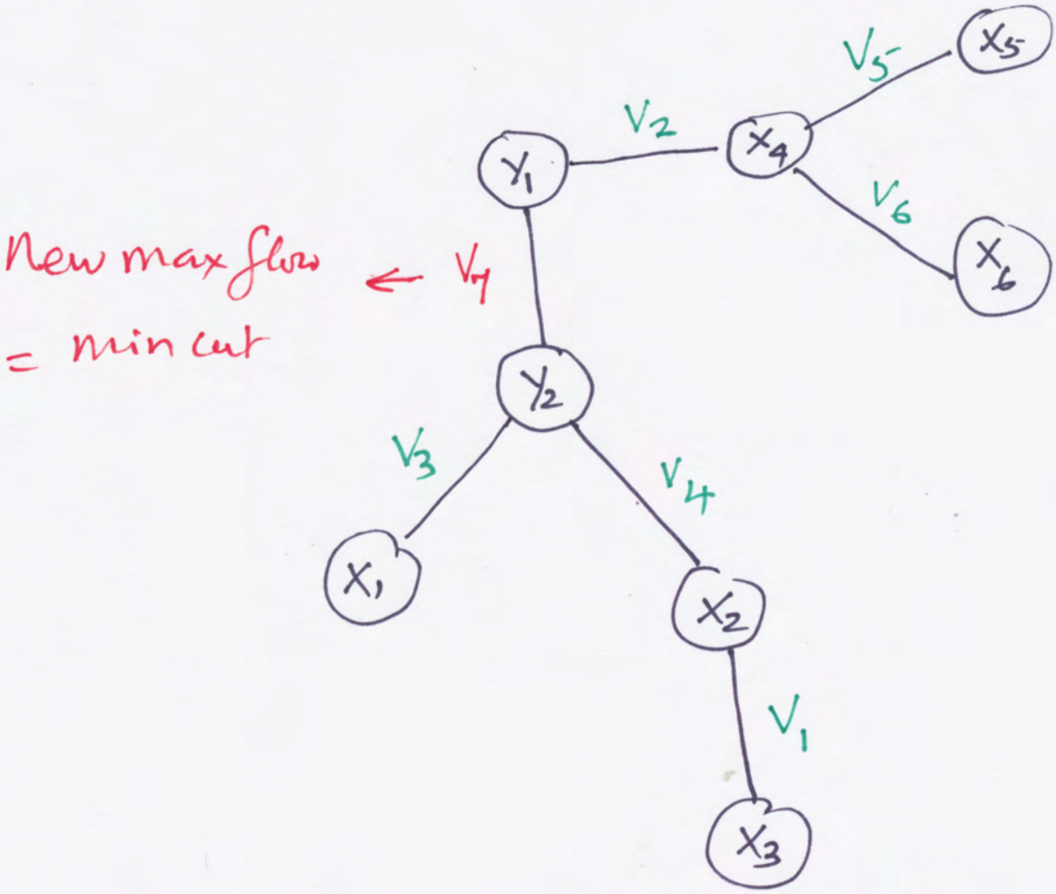


Suppose now we select a pair of nodes in Y for the next step.

When Y is removed from the above diagram we get 3 Connected pieces: $X_2 \cup X_3$, X_1, $X_4 \cup X_5 \cup X_6$

Each of these is (separately) Condensed to create the new Condensed network. Y is <u>not</u> Condensed.

Suppose this splits Y into Y_1 and Y_2



$(X_4 X_5 X_6)$ is on the side of Y_1

(X_1), and (X_2, X_3) are on the side of Y_2.

This process is repeated until all sets are singleton node sets. At this time we have the spanning tree that is "equivalent" to the original network. Now $v_{ij} = \min\limits_{(k,l) \in P_{ij} \in T} v_{k,l} \quad \forall i,j$.

Proof of this follows.

Consider two nodes x and y. In the final tree, removal of any edge partitions the nodes with x and y on opposite sides. Moreover, the value of the edge equals the cut corresponding to this partition. This cut, therefore, separates x from y. $\therefore v_{x,y} \leqq$ value of this cut.

if this edge represented by this cut is on the Unique path in the tree from x to y.

$\therefore v_{x,y} \leqq v_{k,l} \quad (k,l) \in P_{i,j} \in T. \ \forall i,j$

(I) $\therefore v_{x,y} \leqq \min v_{k,l} \ ; (k,l) \in P_{ij} \in T \quad \forall i,j$

Now we show the opposite inequality. For this purpose we need a lemma.

Lemma: At any stage of the Construction, if there is an edge Connecting sets X and Y in the tree, with value v $\exists \ x \in X, \ y \in Y$ such that $v(x,y) = v$

and the cut obtained by removing this edge is a min-cut in the original graph between x and y.

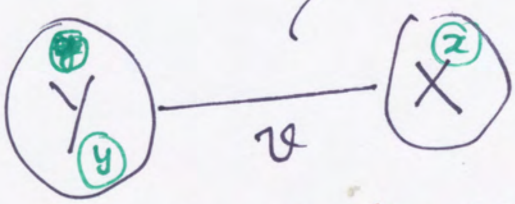
Proof: By induction.

BASIS CASE: At the first step, this is "clearly" true by construction.

Suppose by way of Induction Hypothesis, it is true upto some point. We now select two nodes from a set say Y as in previous diagrams.

one of the edges

Before

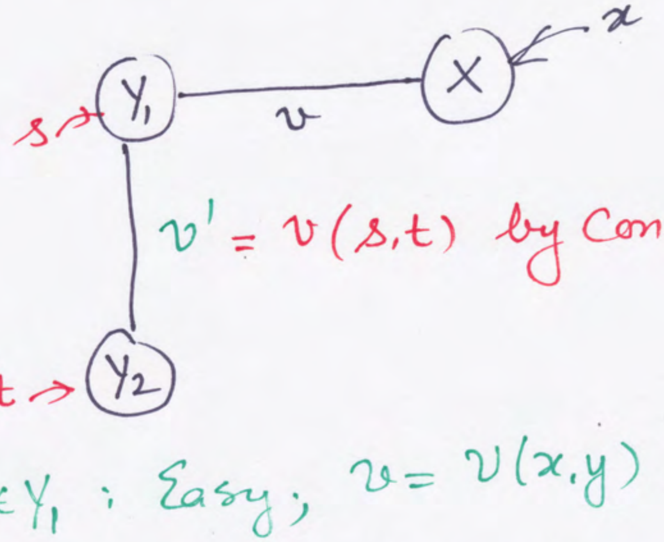


Let $v = v(x, y)$

Let $s, t \in Y$ be the next pair selected.

Suppose Y splits into Y_1, Y_2 with X on the side of Y_1

After



$v' = v(s, t)$ by construction.

Case 1 $y \in Y_1$: Easy; $v = v(x, y)$ as before.

Case 2 $y \in Y_2$ (more difficult case.

Claim : $v = v(s, x)$ in this case.

Pf : x and s are on the same side of s-t mincut that gave rise to v'.

$\therefore$ When solving the ~~s,t~~ s, x problem, Condensing Y_2 does not change max-flow by Lemma shown before.

Let $v(i,j)$: represent max flow in Original network

$\bar{v}(i,j)$: represent max flow in Y_2 Condensed network.

We have

$$\bar{v}(x,s) = v(x,s)$$

$$\bar{v}(x,y) \geqq v(x,y) = v$$

$$\bar{v}(y,t) = \infty$$

$$\bar{v}(t,s) \geq v(t,s) = v'$$

$$v(x,s) = \bar{v}(x,s) \geqq \mathrm{Min}\left[\bar{v}(x,y), \bar{v}(y,t), \bar{v}(t,s)\right]$$

$$= \mathrm{Min}\left[\underset{v}{\underset{\parallel}{\bar{v}(x,y)}}, \underset{v'}{\underset{\parallel}{\bar{v}(t,s)}}\right]$$

But the cut with capacity v' (in this case) also separates x, and y.

$$\therefore v' \geqq v(x,y) = v$$

$$\therefore \text{Min}\,[v', v] = v.$$

$$\therefore v(x,s) \geqq v$$

But this cut with capacity v also separate x and s in this case.

$$\therefore v(x,s) \leq v$$

$$\therefore v(x,s) = v \quad \text{as claimed.}$$

This completes proof of claim.

$\therefore$ Each edge in the final graph represent max flow (<u>and</u> min-cut) for the nodes that form this edge. $\therefore$ Flow values are correct for each pair connected by an edge in the final tree.

$$\therefore v(i,j) \geqq \text{Min}\,[v(i,k), v(k,l), \ldots v(x,j)] \quad \text{for } \forall i,j \qquad \text{(II)}$$

(I) and (II) give the result stated at the beginning.

Hence, the number of maximum flow problems done is $|V|-1$, ~~each~~ some of which are on small graphs (because of condensing)

Condensing is essential to preserve the non-crossing nature of the cuts.

This tree is not only "flow-equivalent" but also cut preserving. and hence is called a Cut-tree equivalent to the original problem.

This completes the analysis problem

Now we turn to the Design or Synthesis problem in the next section.