CS6381 : Lecture #9 : 9.2.2020

Minimum Cost Flows & Related topics :

Statement of the problem

Input : A directed graph $G = [V; E]$; real numbers (integers) q_i ; $i \in V$; l_{ij}, u_{ij}, c_{ij} for each $(i,j) \in E$, $[u_{ij} \geq l_{ij}\ \forall (i,j) \in E]$; ~~fia~~

Desired output : flows $f_{ij}\ \forall (i,j) \in E$ satisfying

$$l_{i,j} \leq f_{i,j} \leq u_{ij} \quad \forall (i,j) \in E$$

$$\sum_j f_{i,j} - \sum_j f_{j,i} = q_i \quad i \in V$$

$$\text{Min} \sum_j \sum_i c_{ij} f_{ij}$$

l_{ij} : lower bound
u_{ij} : upper bound
q_i : external flow
c_{ij} : Cost / unit

Special Cases : Shortest path : (assuming no negative cycles)

~~$u_{ij} = 0$~~ ; ~~f_{ij}~~ $l_{ij} = 0\ \forall (i,j) \in E$.,

$$q_i = \begin{cases} 1 & i = s \\ 0 & i \neq s, t \\ -1 & i = t \end{cases}$$

/ $u_{ij} = 1$ works
$u_{ij} = \infty$ works assuming no neg. cycles.

Special Cases (contd)

Maximum flow

Add an edge (t, s):

$$c_{ij} = 0 \quad \forall (i,j) \neq (t,s)$$

$$c_{t,s} = -1 \; ;$$

Transportation Problem:

$$\sum_j x_{i,j} = a_i \qquad i = 1 \dots m$$

$$\sum_j x_{i,j} = b_j \qquad j = 1 \dots n$$

$$x_{i,j} \geqq 0; \quad \begin{array}{l} i = 1 \dots m \\ j = 1 \dots n \end{array}$$

$$\text{Min} \sum_j \sum_i c_{i,j} x_{i,j}$$

When $m = n$ and $a_i = b_j = 1 \;\forall i, j$ problem is Known as The Assignment Problem (Also bipartite matching problem)

We now describe the basis for a "new" type of algorithm based on Linear Programming duality concepts.

Linear Programs & Duality Theory:

See my lecture notes under Linear Programming or Optimization. (In particular, section on "Duality Theory".)

Linear program:

This is an optimization problem with "linear" constraints and a linear function to be maximized (minimized). [Normally, we also assume that the number of variables and constraints are "finite"]. Hence, it is a problem each of whose constrains [m in total] look like

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \; R_i \; b_i \qquad i = 1 \ldots m$$

where $R_i \in \{ \geqq, \leqq, = \}$

and each variable x_i satisfies the relation

$$x_j \; R'_j \; 0 \qquad j = 1 \ldots n$$

when $R'_j \in \{ \geqq, \leqq,$ no relation to 0 (could be $+, -,$ or 0) $\}$

And the "goal" (objective function) is to max/min

$$\sum_{j=1}^{n} c_j x_j$$

$\{a_{i,j}\}, \{b_i\}, \{c_j\}$ are input data as are R_i, R'_j

There are several "standard form" for LP. For the purposes of Duality Theory:

$$\min cx : Ax \geqq b, x \geqq 0 \qquad \text{(P)}$$

is the one normally used.

$$\min \sum_{j=1}^{n} c_j x_j$$

$$y_i \geqq 0 ; \quad \text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \geqq b_i , \quad i = 1 \dots m$$

$$x_j \geqq 0 ; \quad j = 1 \dots n$$

A standard form: Every LP can be written in this form.

"The" Dual of (P) is another LP:

$$\max b^t y : A^t y \leqq c^t ; \quad y \geqq 0$$

$$\Updownarrow$$

(D)

$$\max \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \leqq c_j \quad j = 1 \dots n$$

$$y_i \geqq 0 \quad i = 1 \dots m$$

Which one is (P) and which is (D) depends on our "interest". Problem of interest is (P)
"The Other" is (D)

First we will demonstrate that this is indeed
"a standard" form. and show the relationship of the dual to the "general" LP.

General LP.

$$
\begin{array}{c|c|c|c|cc}
 & x^1 \geq 0 & x^2 \leq 0 & x^3 \text{: unrestd.} & & \\
\hline
 & A_{11} & A_{12} & A_{13} & \geq & b^1 \\
 & A_{21} & A_{22} & A_{23} & \leq & b^2 \\
 & A_{31} & A_{32} & A_{33} & = & b^3 \\
\hline
\text{Min} & c^1 & c^2 & c^3 & &
\end{array}
$$

Transformation to standard form.

Steps: Multiply by -1: inequalities that go $\leq$

Replace $=$ by $\geq$ & $\leq$

Replace unrestd variables by difference of two nonnegative variables

i.e $x^3 \text{(unrestd)} = x^3_+ - x^3_-$

$$
\begin{array}{ccc}
\| & & \\
8 & \longrightarrow & (8,\ 0) \\
-8 & \longrightarrow & (0,\ +8)
\end{array}
$$

This results in:

$$
\begin{array}{cccccc}
x^1 \geq 0 & -x^2 \geq 0 & x^3_+ \geq 0 & x^3_- \geq 0 & & \\
\left[\begin{array}{c|c|c|c}
A_{11} & -A_{12} & A_{13} & -A_{13} \\ \hline
-A_{21} & -A_{22} & -A_{23} & +A_{23} \\ \hline
A_{31} & -A_{32} & A_{33} & -A_{33} \\ \hline
-A_{31} & A_{32} & -A_{33} & A_{33}
\end{array}\right. & & & & \begin{array}{c} \geq \\ \geq \\ \geq \\ \geq \end{array} & \left[\begin{array}{c} b^1 \\ \hline -b^2 \\ \hline b^3 \\ \hline -b^3 \end{array}\right]
\end{array}
$$

$$
\text{Min} \quad \left[\begin{array}{c|c|c|c} c^1 & -c^2 & c^3 & -c^3 \end{array}\right]
$$

(P)

Which is in standard form. Its dual is

$$
\begin{array}{cccccc}
y^1 \geq 0 & \tilde{y}^2 \geq 0 & \tilde{y}^3_+ \geq 0 & \tilde{y}^3_- \geq 0 & & \\
\left[\begin{array}{c|c|c|c}
A_{11}^t & -A_{21}^t & A_{31}^t & -A_{31}^t \\ \hline
-A_{12}^t & -A_{22}^t & -A_{32}^t & A_{32}^t \\ \hline
A_{13}^t & -A_{23}^t & A_{33}^t & -A_{33}^t \\ \hline
-A_{13}^t & A_{23}^t & -A_{33}^t & A_{33}^t
\end{array}\right] & & & & \begin{array}{c} \leq \\ \leq \\ \leq \\ \leq \end{array} & \left[\begin{array}{c} (c^1)^t \\ \hline (-c^2)^t \\ \hline (c^3)^t \\ \hline (-c^3)^t \end{array}\right]
\end{array}
$$

Combine

$$
\text{Max} \quad \left[\begin{array}{c|c|c|c} (b^1)^t & (-b^2)^t & (b^3)^t & (-b^3)^t \end{array}\right]
$$

Reversing Steps used in getting Standard form.

Let $y^3 = y^3_+ - y^3_-$: unrested

$y^2 = -\tilde{y}^2 \leq 0$

Combining $\geq, \leq \rightarrow =$

We get

$$
\begin{array}{ccc|c|c}
y^1 \geq 0 & y^2 \leq 0 & y^3 : \text{unrested} & & \\
A^t_{11} & A^t_{21} & A^t_{31} & \leq & (c^1)^t \\
A^t_{12} & A^t_{22} & A^t_{32} & \geq & (c^2)^t \\
A^t_{13} & A^t_{23} & A^t_{33} & = & (c^3)^t \\
\end{array}
$$

Max $\left[(b^1)^t \mid (b^2)^t \mid (b^3)^t \right]$

This is "the process" of getting dual to General LP.

Now to the "reason" for looking at dual and relationship between (P) and (D).

Theorem 1. (Weak Duality Theorem)

Let x^0 be any feasible solution to (P) and let y^0 be any " " " (D).

Then $c x^0 \geqq b^t y^0$

Corollary: If x^0 is feas. to (P) and y^0 is feas. to (D) such that $c x^0 = b^t y^0$, then these are "optimal" to (P) and (D) resp.

Pf: (Theorem 1):

x^0 feas to (P) $\Leftrightarrow$ $A x^0 \geqq b$, $x^0 \geqq 0$. $\Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j \geqq b_i$; $i = 1 \ldots n$

y^0 feas to (D) $\Rightarrow$ $y^0 \geqq 0$. y_i^0 $\to \sum b_i y_i^0$

$\sum_{i=1}^{m} y_i^0 \sum_{j=1}^{n} a_{ij} x_j^0$ $\therefore (y^0)^t A x^0 \geqq (y^0)^t b = b^t y^0$

$\sum_{j=1}^{n} x_j^0 \sum_{i=1}^{m} a_{ij} y_i^0$ Similarly $(x^0)^t A y^0 \leqq (x^0)^t c^t = c x^0$.

Combining we get the result.

Note $(y^0)^t A x^0 = (x^0)^t A y^0$

Theorem 2 (Strong Duality) If both (P) and (D) are feasible, then they both have optimal solutions and $c x^* = b^t y^*$ for all opt. solutions x^*, y^*.

Theorem 3: (Weak) Complementary Slackness Theorem)

Let x^* be any optimal solution to (P) and

y^* .. .. .. (D).

Then,

$$\left.\begin{array}{l} \sum_{j=1}^{n} a_{ij} x_j^* > b_i \Rightarrow y_i^* = 0 \\ \sum_{j=1}^{n} a_{ij} x_j^* = b_i \Leftarrow y_i^* > 0 \\ \\ \sum_{i=1}^{m} a_{ij} y_i^* < c_j \Rightarrow x_j^* = 0 \\ \sum_{i=1}^{m} a_{ij} y_i^* = c_j \Leftarrow x_j^* > 0 \end{array}\right\} \quad (**)$$

Moreover, ~~if for~~ any pair (x^*, y^*) that are feas. to (P) and (D) respectively that satisfy $(**)$ are also optimal to (P) and (D) respectively.

It is this result that is basis for algorithms to solve (P) and (D) simultaneously.

These are algorithms based on Duality Theory of Linear Programming.