LINEAR PROGRAMMING AND EXTENSIONS

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Chapter 1

Ratio Optimization

The main problem considered here is of the form $\min_{x \in S}[\frac{f(x)}{g(x)}]$. Throughout this discussion, we assume that $g(x) \neq 0 \; \forall \; x \in S$. If S is convex, then this is the same as assuming that $g(x) > 0 \; \forall \; x \in S$ or $g(x) < 0 \; \forall \; x \in S$. We may assume that the former holds without loss of generality. If f and g are affine and S is a convex polyhedral set, then the problem is known in the literature as a *linear fractional program (LFP)*. It is this problem that we shall address mainly in this chapter, although the techniques are also applicable to some nonlinear versions. We begin with a precise statement of the problem:

1.0.1 Problem I

$$\max \frac{cx + \alpha}{dx + \beta} : Ax \leq b; x \geq 0.$$

Note: The problem is not well defined if

$$\exists \; x^0 \in S \ni Ax^0 \leq b; x^0 \geq 0; dx^0 + \beta = 0$$

Since the set of feasible solutions is a convex set, we must have either

$$dx + \beta > 0 \; \forall \; x \in S$$

or

$$dx + \beta < 0 \; \forall \; x \in S$$

Without loss, we will assume from now on that $dx + \beta > 0 \; \forall \; x \in S$. (Please note that this can be checked by using LP.)

There are basically four approaches to this problem in the literature: (i)Isbell-Marlow methods; (ii)Charnes-Cooper methods; (iii)Dinkelbach-Jagannathan approach; and (iv)Charnes-Martos methods. We shall discuss all the methods along with relative merits of each.

1.1 Charnes-Cooper Approach

References

1. Charnes & Cooper: "Nonlinear Power of Adjacent Extreme Point Methods in Linear Programming", *Econometrica*, 25 (1956) pp 132-153.

2. Charnes & Cooper: "Programming with Linear Fractional Functionals", *Naval Research Logistics Quarterly*, 9 (1962) pp 181-186.

   This work deals primarily with LFP. The main idea is to convert the problem to the following LP in

   which t is a scalar:

1.1.1 Problem II

$$\max\ cy + \alpha t :\ Ay - bt \leq 0;\ dy + \beta t = 1;\ y \geq 0;\ t \geq 0$$

Facts:

1. If x^0 is feasible to I, then $(y^0,\, t^0)$ is feasible to II where

$$t^0 = \frac{1}{dx^0 + \beta};\ y^0 = x^0 t^0;\ cy^0 + \alpha t^0 = t^0(cx^0 + \alpha).$$

2. If $(y^0,\, t^0)$ is feasible to II and $t^0 > 0$, then $x^0 = \frac{y^0}{t^0}$ is feasible to I and $cx^0 + \alpha = \frac{cy^0 + \alpha t^0}{t^0}$.

3. If $(y^*,\, t^*)$ is optimal for II and $t^* > 0$, then $x^* = \frac{y^*}{t^*}$ is optimal for I.

4. II unbounded and I is feasible $\Longleftrightarrow I$ unbounded. Indeed $\exists$ a ray on which each is unbounded

   Proof: If II is unbounded, then

   $\exists\, y^0, t^0 \ni Ay^0 - bt^0 \leq 0;\ dy^0 + \beta t^0 = 0;\ cy^0 + \alpha t^0 > 0;\ y^0 \geq 0;\ t^0 \geq 0$

   $t^0 = 0$; else $x^0 = \frac{y^0}{t^0}$ satisfies the relations $Ax^0 = b$; $x^0 \geq 0$; $dx^0 + \beta = 0$. This contradicts the assumption that $\exists$ no such $x \in S$. Hence the conditions are equivalent to the relations:

$$Ay^0 \leq 0; y^0 \geq 0;\ dy^0 = 0;\ cy^0 > 0$$

   Let x° be any feasible solution to I. Consider $x(\theta) = x^{\circ} + \theta y^{\circ}$. Clearly $x(\theta)$ is feasible to I $\forall\ \theta \geq 0$ and $\frac{cx(\theta)+\alpha}{dx(\theta)+\beta} = \frac{cx^{\circ}+\alpha+\theta cy^0}{dx^{\circ}+\beta} \to \infty$ as $\theta \to \infty$. Conversely, if I is unbounded, then for

$$\forall\ M\ \ \exists\ \ x(M) \ni\ \ Ax(M) \leq b;\ x(M) \geq 0; \frac{cx(M)\ + \alpha}{dx(M) + \beta} >\ M$$

Consider the sequence $(y(M), t(M))$ where $t(M) = \frac{1}{dx(M)+\beta}$ and $y(M) = x(M)t(M)$. Each member of this sequence is feasible to II and $cy(M) + \alpha t(M) = \frac{cx(M)+\alpha}{dx(M)+\beta} > M$. Hence II is also unbounded.

5. If II has optimal solutions and in every optimal solution to II $t = 0$, and I is feasible, then I has a supremum which is not attained, and this case is called *the asymptotic* case. Further, it is possible to produce ϵ-optimal solutions $\forall\ \epsilon > 0$. Also, $\exists$ a ray on which the value tends to the supremum.

   Proof: Since II has optimal solutions by the hypothesis, I is not unbounded. If I has an optimal solution x^0, then by (1) above, $t^0 = \frac{1}{dx^0+\beta}$, $y^0 = x^0 t^0$ is an optimal solution to II with $t^o > 0$; this is a contradiction to the assumption. Thus, I is asymptotic and has a supremum, w^*, which is not achieved by any solution. Let $(y^*, 0)$ be an optimal solution to II. Hence,

$$Ay^* \leq 0; y^* \geq 0; dy^* = 1$$

   Claim: $cy^* = w^*$
   $(i) cy^* \geq w^*$: Let x^o be an ϵ-optimal solution to I and let $\frac{cx^0 + \alpha}{dx^0} + \beta \geq w^* - \epsilon$. Consider (y^0, t^0) with the same meanings for these as before. This is feasible to II and

$$cy^0 + \alpha t^0 = \frac{cx^0 + \alpha}{dx^0 + \beta} \geq w^* - \epsilon$$

   Since this is true, $\forall\ \epsilon > 0$, (i) follows.

   $(ii) cy^* \leq w^*$: Suppose not; $cy^* > w^*$. Consider $x(\theta) = x^0 + \theta y^*$ for $\theta \geq 0$. Clearly this is feasible to I.

$$\frac{c(x^0 + \theta y^*) + \alpha}{d(x^0 + \theta y^*) + \beta} = \frac{cx^0 + \alpha + \theta cy^*}{dx^0 + \beta + \theta} \to cy^* \text{ as } \theta \to \infty$$

   and θ can be chosen so that this value is greater than w^*. But this would lead to a contradiction. Hence, the claim. The ϵ-optimal solution promised before is produced by choosing an appropriately large value for θ in $x(\theta) = x^0 + \theta y^*$.

6. If $[Ax \leq b;\ x \geq 0 \Longrightarrow dx + \beta = 0]$, and $[T = \{x : Ax \leq b;\ x \geq 0\} \neq \phi]$, then II is infeasible.

   Proof: Clearly $\exists$ no solution to II with $t > 0$. Thus, suppose $\exists\ y \ni Ay \leq 0;\ y \geq 0;\ dy = 1$. Let $x^0 \in S$. $d(x^0 + y) + \beta > 1 > 0$ which is a contradiction to the assumption that $dx + \beta \equiv 0\ \forall\ x \in S$.

1.1.2 Advantages

1. This approach shows that LFP is polynomially reducible to LP and, hence, is polynomially solvable.
2. One can derive duality results by using similar results for LP and translating them.
3. All the possibilities are clearly indicated.

Disadvantages:

1. Changes the constraint set; any special structure that might have been used to advantage is lost. For example, if the original set of constraints were that of the transportation problem we would lose that.

Some very fundamental relationships between the two problems should be discussed before we go on.

Theorem 1 *Extreme points of* $T \equiv \{x : Ax \leq b;\ x \geq 0\}$ *are in one-to-one correspondence with extreme points of* $Q \equiv \{(y,t) : Ay - bt \leq 0;\ dy + \beta t = 1;\ y \geq 0;\ t \geq 0\}$ *with* $t > 0$*. We assume that* $x \in T \implies dx + \beta > 0$.

Proof: (i) Let x^0 be an extreme point of T. Let $t^0 = \frac{1}{dx^0+\beta}$ and $y^0 = x^0 t^0$. Clearly $(y^0, t^0) \in Q$. Suppose it is not an extreme point of Q. Then $(y^0, t^0) = \gamma(y^1,\, t^1) + (1-\gamma)(y^2,\, t^2)$ for some points $(y^j,\, t^j) \in Q$ for $j = 1,2$ and $\gamma \in (0,1)$. Since $t^0 > 0$, at least one of t^1, t^2 must be positive.
Case (i): $t^1 > 0$; $t^2 > 0$
Let $x^i = \frac{y^i}{t^i}$, $i = 1,2$; then

$$x^0 = \frac{\alpha t^1}{\alpha t^1 + (1-\alpha)t^2}x^1 + \frac{(1-\alpha)t^2}{\alpha t^1 + (1-\alpha)t^2}x^2$$

This is a contradiction to the hypothesis that x^o is an extreme point of T.
Case $(ii) : t^1 = 0$
Let $x^2 = \frac{y^2}{t^2}$. Then,

$$\begin{aligned} x^0 &= \frac{(1-\alpha)x^2t^2 + \alpha y^1}{(1-\alpha)t^2} \\ &= x^2 + \frac{\alpha}{(1-\alpha)t^2}y^1 \\ &= \frac{1}{2}(x^2 + \frac{\alpha}{2(1-\alpha)t^2}y^1) + \frac{1}{2}(x^2 + \frac{3\alpha}{2(1-\alpha)t^2}y^1) \qquad (1.1) \end{aligned}$$

This again is a contradiction to the assumption that x^o is an extreme point of T. Hence an extreme point of T corresponds to one of Q.

To show the converse, suppose that $(y^0,\ t^0)$ is an extreme point of Q with $t^0 > 0$. Then $x^0 = \frac{y^0}{t^0}$ is an extreme point of T.
Proof: Suppose not. Let $x^0 = \alpha x^1 + (1-\alpha)x^2$. Let $t^i = \frac{1}{dx^i+\beta} > 0$; and $y^i = t^i x^i$ for $i = 1,2$. Hence $(y^0, t^0) = \frac{\frac{\alpha}{t^1}(y^1,t^1)+\frac{(1-\alpha)}{t^2}(y^2,t^2)}{\frac{\alpha}{t^1}+\frac{1-\alpha}{t^2}}$. But this is a contradiction to the assumption that $(y^0,\ t^0)$ is an extreme point of Q. Hence x^0 is an extreme point of T.

Theorem 2 *Extreme points of Q with $t = 0$ correspond to extreme rays of T with $dx > 0$.*

Proof: Let $(y, 0)$ be an extreme point of Q. Clearly, $dy > 0$ and, hence, $y \neq 0$. Let $x = \frac{y}{e^t y}$. $[Ay \leq 0,\ y \geq 0,\ dy > 0,\ y$ extreme point of $Q] \Longrightarrow [Ax \leq 0,\ x \geq 0,\ dx > 0,\ e^t x = 1]$, x is an extreme point of this system and, hence, an extreme ray of T.

An extreme ray x of T corresponds to an extreme point of $T' \equiv \{x : Ax = 0;\ x \geq 0;\ e^t x = 1\}$. If $dx > 0$, then $y = \frac{x}{dx}$ is an extreme point of Q with $t = 0$ and $dy = 1 > 0$.

Theorem 3 *Extreme rays of Q correspond to extreme rays of T with $dx = 0$.*

Proof: Let $(y^0,\ t^0)$ be an extreme ray of Q. Hence $Ay^0 - bt^0 \leq 0$; $dy^0 + \beta t^0 = 0$; $y^0 \geq 0$; $t^0 \geq 0$; $(y^0,\ t^0) \neq 0$. If $t^0 > 0$, then $x^0 = \frac{y^0}{t^0} \in T$ and $dx^0 + \beta = 0$; a contradiction. Thus, $t^0 = 0$. Hence, $y^0 \neq 0$; let $x^0 = \frac{y^0}{e^t y^0}$. $dx^0 = 0$. Also, $x^0 \geq 0$; $Ax^0 \leq 0$; $e^t x^0 = 1$. It is easy to show that x^0 is an extreme point of this set and, hence, an extreme ray of T by contradiction. Hence, the result.

These theorems and their proof techniques apply to the pair $\{x : Ax = b;\ x \geq 0\}$ and $\{y, t : Ay - bt = 0;\ y \geq 0;\ t \geq 0\}$ as well.

1.2 Isbell-Marlow Methods

References:

1. J.Isbell & W.Marlow: "Attrition Games", *Naval Research Logistics Quarterly*, 3, (1956) pp 71-94 (see 82 in particular).

In order to develop these methods, we need to study properties of ratio functions first.
Def: A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be a *quasimonotone* differentiable function if: (i) it is differentiable, and (ii) $f(x^1) > f(x^2) \Longrightarrow (x^1 - x^2)\nabla f(x^0) > 0\ \forall\ x^0 \in [x^1,\ x^2]$ and $\forall\ x^1,\ x^2$, where $[x^1, x^2]$ is the line segment joining these two points.

Lemma 4 *Let $f(x)$ be as in LFP. Then f is a quasimonotone differentiable function.*

Proof: ∇f, in our case, is given by the relation:

$$\nabla f(x) = \frac{(dx+\beta)c - (cx+\alpha)d}{(dx+\beta)^2}$$

Suppose $\frac{cx^1+\alpha}{dx^1+\beta} > \frac{cx^2+\alpha}{dx^2+\beta}$ and $x^0 \in [x^1,\, x^2]$. Then, $\exists\ \theta \in [0,1] \ni x^0 = \theta x^1 + (1-\theta)x^2$. We want to show that:

$$(dx^0+\beta)c(x^1-x^2) - (cx^0+\alpha)d(x^1-x^2) > 0 \tag{1.2}$$

If we let $a = cx^1+\alpha$; $b = dx^1+\beta$; $g = cx^2+\alpha$; and $h = dx^2+\beta$, then

$$cx^0+\alpha = \theta a + (1-\theta)g; dx^0+\beta = \theta b + (1-\theta)h; (a/b) > (g/h) \tag{1.3}$$

Showing (1.2) is equivalent to showing (1.4) where

$$[\theta b + (1-\theta)h][a-g] - [\theta a + (1-\theta)g][b-h] > 0 \tag{1.4}$$

To show (1.4) we note that in (1.4)

$$\begin{aligned} LHS &= [\theta ab + (1-\theta)ah - \theta bg - (1-\theta)gh] - [\theta ab + (1-\theta)bg - \theta ah - (1-\theta)gh] \\ &= ah - bg \\ &> 0 \end{aligned} \tag{1.5}$$

as required.

Consider the problem: $\min_{x\in S}\ f(x)$ where S is a convex set and f is a differentiable quasimonotone function. Then the following algorithm (that starts with a feasible solution) produces an improving sequence of feasible solutions:

Algorithm: Start with $x^0 \in S$. Solve $\min_{x\in S} [\nabla f(x^0)x]$. If x^0 solves this problem, it is an optimal solution to the original problem. If not, a new improved solution x^1 is obtained and the algorithm continues.

If S is a convex polyhedral set, then we may start the algorithm at an extreme point x^0 of S. In this case, the algorithm solves an LP at each step and, hence, each solution in the sequence may be taken to be an extreme point of S. If x^0 solves the LP, we are done since it solves the original problem. If a better extreme point is visited by the algorithm, then the last such point (in case of unboundedness) is the next point for the algorithm. Thus, the algorithm will terminate finitely in this case assuming that S is bounded as well. If, at some stage of the algorithm, no adjacent extreme point of x^0 is better and the subproblem is unbounded, then, in general, we are stuck. There are three possibilities in this case: (i) $\exists$ finite optimal solution to the original problem; (ii) the original problem is unbounded; and (iii) a finite lower bound that is not achieved – asymptoticness. We will show later how to detect case (i); detecting

(ii) or (iii) will, in general, be more difficult for general f. *An interesting open question: Is there a polynomially bounded algorithm for this problem given that there is one for LP?*

Now we continue the discussion of the special case of LFP. Isbell and Marlow give a special algorithm for this case; we first describe their algorithm and then show it is a special version of the above.

Isbell-Marlow Algorithm: **Step 0**: Select $x^0 \in S$ (Use Phase I of LP for example); $k = 0$

Step 1: Let $w^k = \frac{cx^k+\alpha}{dx^k+\beta}$

Step 2: Solve the LP: $\min_{x\in S}[(c - w^k d)x]$; if x^k solves this LP, stop — x^k is the required optimal solution to the original problem. If not, let x^{k+1} be an optimal solution to this problem, if one exists (Isbell and Marlow assume that S is bounded throughout their analysis) and go to Step 1 with k increased by 1.

End

If $f(x) = \frac{cx+\alpha}{dx+\beta}$, then

$$\nabla f(x^n) = \frac{[(dx^n + \beta)c - (cx^n + \alpha)d]}{(dx^n + \beta)^2} = \frac{[c - \frac{(cx^n+\alpha)}{(dx^n+\beta)}d]}{dx^n + \beta}$$

Hence, $\max_{xx\in S}[(c - w^n d)x]$ is equivalent to $\max_{x\in S}[\nabla f(x^n)x]$. Let $S = \{x : Ax = b;\ x \geq 0\}$ and let $dx + \beta > 0\ \forall\ x \in S$. If the LP above is unbounded at some step,

$$\exists\, x^0 \ni x^0 \geq 0; Ax^0 = 0; (c - w^n d)x^0 > 0$$

Clearly, $dx^0 \geq 0$; else $d(x + \theta x^0) + \beta < 0$ for sufficiently large θ. If $dx^0 = 0$, then the original problem is unbounded; for $x + \theta x^0$ is feasible to the original problem $\forall\ \theta \geq 0$, and $\frac{c(x+\theta x^0)+\alpha}{d(x+\theta x^0)+\beta} = \frac{cx+\alpha+\theta cx^0}{dx+\beta} \to \infty$ as $\theta \to \infty$. If $dx^0 > 0$, then $w^n < \frac{cx^0}{dx^0}$. Then,

$$\frac{c(x + \theta x^0) + \alpha}{d(x + \theta x^0) + \beta} = \frac{cx + \alpha + \theta cx^0}{dx + \beta + \theta dx^0} \to \frac{cx^0}{dx^0} > w^n \text{ as } \theta \to \infty.$$

In this case, define $w^{n+1} = \frac{cx^0}{dx^0}$ and go to Step 3.

$$\begin{aligned}(c - w^{n+1}d)(x + \theta x^0) &= cx + \theta cx^0 - w^{n+1}(dx + \theta dx^0)\\ &= (c - w^{n+1}d)x + \theta(c - w^{n+1}d)x^0\\ &= (c - w^{n+1}d)x \end{aligned} \qquad (1.6)$$

If the optimal value to $\max_{x\in S}[(c - w^{n+1}d)x] > w^{n+1}\beta - \alpha$, then we use the new value in Step 3. If the problem is unbounded, we get a new ray and its w or the implication that the original problem is unbounded. If optimal value equals $w^{n+1}\beta - \alpha$, then this solution is the required optimal solution to the original problem. If the max value is less, than we have the asymptotic case. $\exists$ no $x \in S \ni \frac{cx+\alpha}{dx+\beta} \geq w^{n+1}$ and $(x + \theta x^0)$ above satisfies the relation: $\frac{c(x+\theta x^0)+\alpha}{d(x+\theta x^0)+\beta} \to w^{n+1}$ as $\theta \to \infty$ and is the asymptotic solution to the original problem.

1.3 Martos-Charnes-Cooper Methods

References

1. Charnes & : "Nonlinear Power of Adjacent Extreme Point Methods in LP",Cooper *Eonometrica*, 25 (1956), pp 132 — 153.

2. Martos, B: "Hyperbolikas Programozas", *Publ. Math. Inst. Hung. Acad. Sci.* V, Ser B, #4, (1960) pp 383 — 406. (English version: *Naval Research Logistics Quarterly* 11, (1964) pp 135 — 155.)

3. Martos, B: "The Direct Power of Adjacent Vertex Programming Methods," *Management Science*, 12, (1965) pp 241-252. Correction: 14 (1967 Nov) pp 255-256.

The fundamental idea is the same as in the simplex method of moving from an extreme point to a better adjacent one until none exists. A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be *quasiconvex* if $\lambda \in [0,1] \implies f(\lambda x^1 + (1-\lambda)x^2) \leq \max[f(x^1), f(x^2)]$. f is said to be *explicitly quasiconvex* if $f(x^1) \neq f(x^2) \implies f(\lambda x^1 + (1-\lambda)x^2) < \max[f(x^1), f(x^2)] \forall \lambda \in (0,1)$. *In other words, the only "flats" that are permitted are at the global minimum.* Explicit quasiconcavity is defined similarly.

Lemma 5 *Let S be a closed convex set and f be a quasiconcave (quasiconvex) function on S. Then, if $\min_{x\in S}[f(x)]$ ($\max_{x\in S}[f(x)]$) is achieved, it is achieved at an extreme point of S (assuming one exists).*

Lemma 6 *Let S be a closed convex set and f be an explicitly quasiconvex (explicitly quasiconcave) function over S. Then, each local minimum (maximum) is a global minimum (maximum).*

Lemma 7 *The function $\frac{cx+\alpha}{dx+\beta}$ is both explicitly quasiconvex and explicitly quasiconcave.*

These lemmas are the basis for the adjacent vertex methods like the simplex algorithm. We will now consider the case of LFP in detail.

Algorithm: Let B be a feasible basis; $\bar{c} = c - c_B B^{-1} A$ and $\bar{d} = d - d_B B^{-1} A$. Suppose x_j is nonbasic. The adjacent vertex produced by increasing x_j maintaining all other nonbasic variables at 0 is given by $\hat{x}_B = x_B + (B^{-1}A)_{.j}\hat{x}_j$; $x_j = \hat{x}_j$ and $x_r = 0 \, \forall \, r$ nonbasic and $r \neq j$. It is easy to show that:

$$\frac{c\hat{x} + \alpha}{d\hat{x} + \beta} = \frac{cx + \alpha + \bar{c}_j \hat{x}_j}{dx + \beta + \bar{d}_j \hat{x}_j}$$

If we let the value corresponding to the basic solution for the basis B be θ_B, the new solution $\hat{x}$ is "better" iff $\bar{c}_j - \theta_B \bar{d}_j < 0$. This is the criterion for entering variable selection. In case the entering variable is x_s and $\bar{a}_{i,s} \leq 0$, there are two cases to consider:

(i) $\bar{d}_s = 0$ — in this case, the problem is unbounded;

(ii) $\bar{d}_s > 0$ — Let $\bar{\theta} = \frac{\bar{c}_s}{\bar{d}_s}$ and repeat the process.

If the last value of θ corresponds to the form $\frac{cx+\alpha}{dx+\beta}$, then we have an optimal solution; if it is of the form (ii) above, then we have the asymptotic case.

1.4 Dinkelbach-Jagannathan Methods

References:

1. W. Dinkelbach: "Die Maximierung eines quotientien zwier linearer funktionen unter linearen nebenbedingungen," *Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gabiete* (1962) pp 141 — 145.

2. R. Jagannathan: "On Some Properties of Programming Problems in Parametric Form Pertaining to Fractional Programming," *Management Science*, 12, (1966) pp 609 — 615.

3. W. Dinkelbach: "On Nonlinear Fractional Programming," *Management Science*, 13 (1967), pp 492 — 498.

This approach rests on the following results.

Lemma 8 *Let* $\lambda^* = \min_{x \in S}[\frac{f(x)}{g(x)}]$, *and* $\theta(\lambda) = \min_{x \in S}[f(x) - \lambda g(x)]$. *Then, the sign of* $(\lambda^* - \lambda)$ *is the same as that of* $\theta(\lambda)$. *(It is assumed that* $g(x) > 0 \, \forall \, x \in S$.*)*

Proof: $\lambda^* - \lambda > 0 \Longrightarrow \frac{f(x)}{g(x)} > \lambda \, \forall \, x \in S$; hence $f(x) - \lambda g(x) > 0 \, \forall \, x \in S$, and hence,$\theta(\lambda) > 0$ if minimum is achieved. If $\lambda^* - \lambda < 0$, $\exists \, x^* \ni \lambda^* = \frac{f(x^*)}{g(x^*)} < \lambda$, and hence, $\theta(\lambda) < 0$. If $\lambda = \lambda^*$, $\exists \, x^* \ni f(x^*) - \lambda g(x^*) = 0$; and $\frac{f(x)}{g(x)} \geq \lambda^* \, \forall \, x \in S$, and hence, $\theta(\lambda) \geq 0 \, \forall \, x \in S$.

Suppose $\min_{x \in S}[\frac{f(x)}{g(x)}]$ is not achieved but $\inf_{x \in S}[\frac{f(x)}{g(x)}] = \lambda^*$. Then, the sign of $\lambda^* - \lambda$ is the same as that of $\inf_{x \in S}[f(x) - \lambda g(x)]$, and this is never achieved. The proof for this is similar to the previous one. If $\min_{x \in S}[\frac{f(x)}{g(x)}]$ is unbounded, then $\min_{x \in S}[f(x) - \lambda g(x)] < 0 \; \forall \; \lambda$. Thus, we have a scheme in some instances to solve $\min_{x \in S}[\frac{f(x)}{g(x)}]$ by solving $\min_{x \in S}[f(x) - \lambda g(x)]$ instead; varying the parameter λ will give us the optimal value λ^* through a binary search on λ. Recall that if x^* solves $\min_{x \in S}[f(x) - \lambda^* g(x)]$, then it also solves the original problem. Now we apply these to LFP.

Lemma 9 *LFP is unbounded iff the LP:* $\min_{x \in S}[(cx + \alpha) - \lambda(dx + \beta)]$ *is unbounded* $\forall \; \lambda$*.*

Proof: We have already shown that LFP is unbounded iff $\exists \; x^0 \ni x^0 \geq 0$; $Ax^0 = 0$; $cx < 0$; $dx^0 = 0$. Also the LP is unbounded for a fixed λ iff $\exists \; x^1 \ni x^1 \geq 0$; $Ax^1 = 0$; $cx^1 - \lambda dx^1 < 0$. For such a fixed extreme ray, x^1 of $\{x : x \geq 0; Ax = b\}$, if $dx^1 > 0$, then, for $\lambda < \frac{cx^1}{dx^1}$, we have $cx^1 - \lambda dx^1 > 0$. Thus, if the LP is unbounded for all λ, we must have an extreme ray $x^2 \ni dx^2 = 0$ and $cx^2 < 0$. The converse is easy.

It is easy to see that if x^* is an optimal solution to LFP with $\frac{cx^* + \alpha}{dx^* + \beta} = \lambda^*$, for $\lambda = \lambda^*$, the LP has a minimum value equal to 0 and x^* is optimal.

Lemma 10 *The LFP is asymptotic iff the LP has an optimal solution with minimum value less than* $0 \; \forall \; \lambda \leq \lambda^*$ *and* $\forall \; \lambda > \lambda^*$ *it is unbounded.*

Proof: LFP asymptotic $\Longleftrightarrow [\exists \; x^0 \ni x^0 \geq 0;\; Ax^0 = 0;\; \lambda^* = \frac{cx^0}{dx^0}]$ and LFP is not unbounded. Hence, $\lambda > \lambda^* \Longrightarrow cx^0 - \lambda dx^0 < 0$; hence, the LP is unbounded for $\lambda > \lambda^*$. Since $\frac{cx+\alpha}{dx+\beta} > \lambda^* \; \forall \; x \in S$, $\lambda < \lambda^* \Longrightarrow \frac{cx+\alpha}{dx+\beta} > \lambda \; \forall \; x \in S$. Hence, the LP value is positive. Hence, minimum exists and is positive. For $\lambda = \lambda^*$, $\forall \; x \in S$, $\frac{cx+\alpha}{dx+\beta} > \lambda$ and, hence, $(cx + \alpha) - \lambda(dx + \beta) > 0$; again maximum exists for the LP and is positive.