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0.0.1 Proofs

1. $x, y \in S \cap T \Longrightarrow x, y \in S$, and hence $\lambda x + (1-\lambda)y \in S$; similarly $\lambda x + (1-\lambda)y \in T$ and hence it is in $S \cap T$. Hence the result. The assertion about $\cap_{\alpha \in A} S_\alpha$ follows in a similar manner.

2. "Clearly" the assertion is true by definition for $n = 2$. Assume that it is true for $n \leq k$. Let $x =_i^k \sum_{=1}^{+1} \lambda_i x^i = \lambda_1 x_1 + (1-\lambda_1)_i^k \sum_{=2}^{+1} \mu_i x_i = \lambda_1 x_1 + (1-\lambda_1) z$ where $\mu_i = \lambda_i/(1-\lambda_1) \geq 0$, and $\sum_{k=2}^{k+1} \mu_i = 1$. (*Please note that we have assumed that $\lambda_1 > 0$; but this is without loss since not all λ_i can be 0*). Since z is a convex combination of k members of S, $z \in S$; hence $x \in S$. *Please note that this proof will not generalize to the infinite case which needs a different kind of proof.*

3. To show that H is a convex set: Let $x, y \in H$. Hence $cx = \alpha$, and $cy = \alpha$. Thus, $\lambda cx + (1-\lambda)cy = c(\lambda x + (1-\lambda)y) = \alpha$. Hence H is convex. H^+, H^{++}, H^-, and H^{--}*are* shown to be convex in a similar manner. *Indeed, this is the most primitive way of showing a set to be convex.*

4. To show that D is convex: Let $x, y \in D$. Hence $\sum(x_i - a_i)^2 \leq \alpha^2$; and $\sum(y_i - a_i)^2 \leq \alpha^2$. Need to show that $\sum[\lambda(x_i + (1-\lambda)y_i) - a_i]^2 \leq \alpha^2$. $\sum[\lambda(x_i + (1-\lambda)y_i) - a_i]^2 = \sum[\lambda(x_i - a_i) + (1-\lambda)(y_i - a_i)]^2. = \lambda^2 \sum(x_i - a_i)^2 + (1-\lambda)^2 \sum(y_i - a_i)^2 + 2\lambda(1-\lambda)\sum(x_i - a_i)(y_i - a_i). = \lambda \sum(x_i - a_i)^2 + (1-\lambda)\sum(y_i - a_i)^2 - \lambda(1-\lambda)\sum[(x_i - a_i) - (y_i - a_i)]^2$ $\leq \alpha^2$.

5. The set of feasible solutions is closed because the limit points of a sequence of points in the set is in the set. The rest is easy.

0.0.2 Theorems

T1 Consider the problem $\min_{y \in \hat{S}} d(x, y)$; note $d(x.y) = \|x - y\|$. Since distance is a continuous nonnegative function and $\hat{S}$

is closed, the minimum is achieved at a point $y^* \in \hat{S}$, and $y^* \neq x$. Consider the halfspace $H^+ = \{y : 2(y^* - x)^t y \geq (y^* - x)^t(y^* + x)\}$. It is easy to show that $y^* \in H^+$ and $x \notin H^+$. For $y \in \hat{S}$, $\lambda y + (1-\lambda)y^* \in \hat{S}$ since S is convex. By the definition of y^*, we have:

$$\|[\lambda y + (1-\lambda)y^*] - x\|^2 = \|y^* + \lambda(y - y^*) - x\|^2 \geq \|y^* - x\|^2$$

This on simplification yields:

$$2\lambda(y - y^*)^t(y^* - x) + \lambda^2(y - y^*)^t(y - y^*) \geq 0 \; M \; \forall \; \lambda \in [0,1]$$

Hence:

$$(y - y^*)^t(y^* - x) \geq 0 > (x - y^*)^t(y^* - x)/2$$

and therefore,
$2(y - y^*)^t(y^* - x) + (y^* - x)^t(y^* - x) > 0$, and hence
$(2y - y^* - x)^t(y^* - x) > 0$, and hence
$2y^t(y^* - x) > (y^* + x)^t(y^* - x)$ which is the required result.
Hence $\hat{S} \subset H^{++}$, and $x \in H^{--}$.

Theorem 1 *Since x is a boundary point of S, $\exists$ a sequence of points $x_\nu \notin \hat{S} \ni \lim\ x_\nu = x$. (Choose each of these to be in smaller and smaller spheres centered at x and from outside $\hat{S}$). For x_ν, $\exists\ a_\nu \ni \|a_\nu\| = 1$, and $a_\nu^t y > a_\nu^t x_\nu \forall y \in \hat{S}$. Let a be a limit point of the sequence $\{a_\nu\}$. (Such points exist since all of these are in a compact set). Then $a^t y \geq a^t x\ \forall\ y \in \hat{S}$ with equality holding for x. This is the required supporting plane.*

Theorem 2 *If S is a closed convex set and T is the intersection of all its supporting halfspaces, then "clearly" $S \subseteq T$. If $x \in T - S$, then by theorem 1, $\exists\ H \ni S \subset H^{++}$ and $x \notin H^{++}$, and this is a contradiction to the definition of T.*

Theorem 3 *4.*

0.0.3 Functions

2 Let $f(x) = \sup\ f_\nu(x)$. Since $f_\nu(\lambda x + (1-\lambda)y) \leq \lambda f_\nu(x) + (1-\lambda)f_\nu(y) \forall \nu$, the result follows.

3 Let $N(x^0)$ be the neighborhood of x^0. If $f(x^0) > f(x^1)$, then $\exists z = \lambda x^0 + (1-\lambda)x^1$ and $z \in N(x^0)$ with $f(z) < f(x^0)$.

0.1 LP Related Results

2 Let x be a basic feasible solution corresponding to the basis B (think of a basis as a set of linearly independent columns of A). By this we mean that $x^{NB} = 0$, and $Bx^B = b$ has a unique solution. If $x = \lambda y + (1-\lambda)z$, then $y^{NB} = z^{NB} = 0$; hence $y^B = z^B = x^B$. Hence $x = y = z$. Hence x is an extreme point.

Conversely, suppose x is an extreme point, and let B be the columns corresponding to positive components of x. Suppose these are dependent; $\exists\ z^B \neq 0 \ni Bz^B = 0$. By letting other components equal to 0 we get $z \neq 0 \ni Az = 0$. Consider the vector $x + \theta z$. Since $x^{NB} = z^{NB} = 0; (x+\theta z)^{NB} = 0\ \forall\ \theta$; also since $Az = 0; A(x+\theta z) = b\ \forall\ \theta$. Since $x^B > 0$, $\exists\ \theta > 0 \ni (x \pm \theta z)^B \geq 0$. This is a contradiction of extremeness of x. Hence the columns of B are independent and hence x is a basic feasible solution.