LINEAR PROGRAMMING AND EXTENSIONS

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LINEAR ALGEBRA

Vector Space:

A *vector space* V over a *field* R is a set of objects called *vectors* together with two binary operations. The first operation is called addition and for every pair of vectors u and v in V this operation satisfies the following conditions:

(i) $u + v$ is a uniquely defined element in V

(ii) $u + v = v + u$

(iii) $u + (v + w) = (u + v) + w$

(iv) $\exists$ an element in V called $0_V \ni u + 0_V = 0_V + u = u \; \forall u \in V$. This element is called the (*additive*) *identity* of V.

(v) $\forall u \in V \; \exists$ a unique element $(-u)$ of V called *the* (*additive*) *inverse of* u that satisfies the relation: $(-u) \ni u + (-u) = 0_V$.

The second operation is called *scalar multiplication* and it satisfies the following conditions:

(vi) αu is an element of $V \; \forall u \in V$ and $\forall \alpha \in R$.

(vii) $\alpha(u + v) = \alpha u + \alpha v \; \forall \alpha \in R$ and $u, v \in V$.

(viii) $(\alpha + \beta)u = \alpha u + \beta u \; \forall \alpha, \beta \in R$ and $\forall u \in V$.

(ix) $\alpha(\beta u) = (\alpha\beta)u \; \forall \alpha, \beta \in R$ and $\forall u \in V$.

(x) $1.u = u$; $0u = 0_V$ where 0 and 1 are the additive and multiplicative identities of the field R and $u \in V$.

The elements of R are called *scalars*. The set of all n-tuples $(a_1, a_2, ..., a_n)$ is called the vector space of n-tuples over R if all the $a_i \in R$. The 0_V for this space is the vector $(0, 0, ..., 0)$. If R is the field of reals then this space is denoted by R^n.

Linear Dependence:

If V is vector space over R and $u^i \in V$ and $\alpha_i \in R; 1 \leq i \leq n$, then $u = \sum_i \alpha_i u^i$ is called a *linear combination* of the u^i. the α's are called the *coefficients*. The set of all linear combinations of a given set of vectors is vector space and is said to be the space *spanned* by this set. If 0 can be expressed as a linear combination of a set of vectors with not all the coefficients equal to zero then this set is said to be *linearly dependent*. A set that is not dependent is said to be *linearly independent*. If $\exists$ a finite set of vectors that spans a vector space then this space is said to be finite dimensional and the cardinality of a minimal set of vectors that spans a space is called the dimension of the space. Such a set of vectors is called a *basis* of the space. Equivalently a set of linearly independent vectors that spans the space is called the basis. It will turn out all these sets have the same cardinality — the dimension of the space. For example, the set of vectors spanned by the unit vectors $e_i = (0, 0, ...1, 0, 0, ...0)$ where the 1 is in the i^{th} place for $i = 1, 2, ..., n$ form a vector space. Clearly, the coefficients in any linear combination with this basis is unique; it turns out the same result is true for any basis. These coefficients are called the coordinates with respect to this basis.

Theorem 1 : *Let V be vector space over a field R. Then:*

1. *If a set $u^i; i = 1, 2, ..., n$ are linearly dependent then $\exists$ an index $k \ni u^k$ is a linear combination of the rest. (Please note that all the coefficients may be equal to zero in a linear combination).*

2. *If V contains a some nonzero element and V is finite dimensional, then it has a basis.*

3. *If u^i, $i = 1, 2, ..., n$ is a basis for V and v^j, $j = 1, 2, ..., r$ is a linearly independent set of vectors, then $r \leq n \exists$ a basis of the form v^1, v^2, ... v^r, u^{i_1}, u^{i_2}, ..., $u^{i_{n-r}}$.*

4. *If V is finite dimensional then any two bases have the same number of elements.*

Proof:(1) Let $\sum c_k u^k = 0_V$; since some $c_k \neq 0$, we can express u^k in terms of the others by the relation $u^k = -\sum_{j \neq k}(c_j/c_k)u^j$.

(2) Since V is finite dimensional $\exists$ a finite set of vectors that span the space. Let this set be X. Let $U = \{u^1, u^2, ..., u^n\}$ be a maximal subset of X that is linearly independent (this is possible because $V \neq \{0_V\}$). Claim that this is a basis for V. For any $x \in X$ with $x \notin U$, the set $\{x, u^1, u^2, ..., u^n\}$ is linearly dependent because U is a maximal independent set in X and hence $\exists c_i$, $0 \leq i \leq n$ satisfying $c_0 x + \sum_i c_i u^i = 0_V$. Clearly, $c_0 \neq 0$; else the set U is dependent. Thus, x can be expressed as a linear combination of the $u's$. Thus, U spans X. Since X spans V, all we have to do is to substitute for elements in $X - U$, the corresponding linear combinations in terms of the $u's$; this would show that U spans V. Hence U is a basis.

(3) Since U is a basis, v^1 can be written as a linear combination of the $u's$. None of the vectors v^i can be 0_V since the set V is linearly independent. Thus, at least one of the coefficients in the linear combination is different from zero. Thus, we can now interchange the roles of this v and the corresponding u. Repeating this procedure yields the desired result.

(4) An immediate consequence of (3).

The number of vectors in any basis is called the dimension of the space except in the case $V = \{0_V\}$ in which case the dimension is 0. If $W \subset V$ is also a vector space over R then W is called a subspace of V.

Theorem 2 : *Any set of $n+1$ vectors in an n dimensional vector space V are linearly dependent.*

Theorem 3 : *If w is a subspace of V then $\dim W \leq \dim V$. Equality holds if $W = V$.*

Inner (*scalar*) *Product* of two vectors u and v (both from the same vector space) is defined usually by the expression $\sum_j u_j v_j$ or the by $\|u\|.\|v\|.\cos\ \theta$

where $\|x\|$ refers to the norm of the vector (or its length) and θ is the angle between the two vectors in the plane defined by them. In general, the inner product is a binary operation defined on the vector space that satisfies the following relations:

(i) $(\alpha u + \beta u,\, v) = \alpha(u, v) + \beta(u, v)$

(ii) $(u,\, u) \geq 0\ \forall u$; and equality holds iff $u = 0_V$.

(iii) $(u,\, v) = \overline{(v, u)}$ where $\bar{x}$ is the complex conjugate of x for all $x \in F$ where F is either the real or complex field.

If R is the field of reals and the inner product is the usual one, then the pair V together with this inner product is called the *Euclidean space*. If u and v are a pair of vectors $\ni (u, v) = 0$ then we say that u and v are perpendicular or orthogonal. $(u, u)^{\frac{1}{2}}$ is called the length or euclidean norm of the vector. A basis all of whose vectors are of length 1 and are pairwise orthogonal is called an orthonormal basis. The unit vectors is an example of this type of basis. If W is a subspace of a vector space V then the set $\{x : (x, u) = 0\ \forall u \in W\}$ is called the orthogonal complement of W relative to V. That this is a vector space is easy to show. The following theorem is often used to produce a set of orthonormal set of vectors spanning a subspace generated by a given set of vectors.

Theorem 4 : *Let $u_1, u_2, ..., u_n$ be a set of linearly independent vectors from a vector space V. Then $\exists$ a set of orthonormal vectors $v_1, v_2, ..., v_n \ni$ the subspace generated by the u's and that generated by the v's is the same.*

Theorem 5 : *The row rank of a matrix equals its column rank.*

Proof: Let row rank be m and let column rank equal n and without loss let us suppose that $m < n$. Let the n linearly independent columns be $A_{.1}, A_{.2}, ..., A_{.n}$. Since the row rank equals $m < n$, let us suppose that the rows $A_{1.}, A_{2.}, ..., A_{m.}$ are linearly independent and hence span the others. Hence $a_{r,j} = \sum_{i=1}^{m} \alpha_i a_{i,j}$ for $\forall r \geq m+1$. Considering only the first m rows of A (let the corresponding submatrix be denoted by $\hat{A}$), since all the (sub)-column vectors are of length m, any basis can contain no more than m vectors. Hence, $\hat{A}_{.j}$, $1 \leq j \leq n$ are linearly dependent. Combining these two arguments we get that $A_{.j}$, $1 \leq j \leq n$ are linearly dependent. Hence the theorem.