LINEAR PROGRAMMING AND EXTENSIONS

R. Chandrasekaran
The University of Texas at Dallas

August 28, 1998

Results in Linear Algebra

Definitions:

1. A *matrix* is a rectangular array of (usually) numbers (real for the purpose of this course). It is customary to specify the size of a matrix by specifying the number of its rows and then the number of its columns. For example, a $m \times n$ matrix has m rows and n columns. A $m \times 1$ matrix is called a *column vector* and a $1 \times n$ matrix is called a *row vector*. If $m = n$ the matrix is called a *square* matrix. The numbers in the array are called the elements of the matrix. For example, $a_{i,j}$ refers to the element of the matrix A in the i^{th} row and the j^{th} column. A square matrix all of whose elements except the diagonal $(a_{i,i})$ are zero and all of whose diagonal elements are equal to 1 is called the (*multiplicative*) *identity* matrix which is denoted by I. A square matrix all of whose rows (columns) are identical to that of the identity matrix except one is called a row (column) *elementary* matrix.

2. Two matrices of the same size (meaning the same number of rows and the same number of columns) can be added. Thus, if A and B are two $m \times n$ matrices, then their sum $C = A + B$ is also a $m \times n$ matrix and $c_{i,j} = a_{i,j} + b_{i,j}$. If A is a $m \times n$ matrix then λA is a $m \times n$ matrix B with $b_{i,j} = \lambda a_{i,j}$.

3. If A is a $m \times n$ matrix and B is a $n \times p$ matrix then they can be multiplied; the requirement is that the number of columns in the first be equal to the number of rows in the second – this asymmetry makes this operation noncommutative. If the product is $C = AB$, then it is a matrix of size $m \times p$ and its element $c_{i,j} = \sum_k a_{i,k}.b_{k,j}$. Thus, not only may $AB \neq BA$, but one of these may not even be defined while the other is. It is easy to see that $A(B + C) = AB + AC$ and $(AB)C = A(BC) = ABC$. A $m \times n$ matrix can be thought of as m row vectors each of length n written one below the other or as n column vectors of size m written one after the other. $A_{.j}$ will represent the j^{th} column of the matrix A and $A_{i.}$ will represent the i^{th} row of A. The j^{th} column of the identity matrix I will be represented by e_j and the column vector all of whose elements are equal to 1 by e. 0 will denote the matrix all of whose entries are 0; the context will determine the dimensions (size) of these matrices. Note $AI = IA = A$. If A is a $m \times n$ matrix then A^t, the *transpose* of A is a $n \times m$ matrix and with i^{th} column of A^t being the same as the i^{th} row of A written vertically; i.e. if $A_{i.} = (\alpha_1, \alpha_2, ..., \alpha_n)$ then $A^t_{.i} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix}$. It is easy to see that

$(A+B)^t = A^t + B^t$ and $(AB)^t = B^t A^t$. If A is $1 \times n$ and B is $n \times 1$ then AB is called the *scalar product* of these two vectors and is a 1×1 matrix (or a number). If A is a $m \times n$ matrix , B is $n \times p$ and C is $n \times q$ then $D = [B, C]$ is a $n \times (p+q)$ matrix, the first p columns of which are the corresponding columns of B and the last q are those of C written in the same order as that in B and C. Also $AD = [AB, AC]$ is a matrix of size $m \times (p+q)$. D is called a partitioned matrix. Similar remarks apply to partitioning along rows. If $C = AB$ then it is easy to see that $C_{i.} = A_{i.}B$ and $C_{.j} = AB_{.j}$.

Linear Dependence:

1. A finite set $a^1, a^2, ..., a^n$ of column vectors of the same size is said to be *linearly dependent* if $\exists$ numbers $\alpha_i; i = 1,\ 2,\ ...,\ n$ not all of which are zero $\ni \sum_{i=1}^{n} \alpha_i a^i = 0$. A set that is not linearly dependent is linearly independent. If a set is dependent and $\alpha_i \neq 0$ then the corresponding vector a^i can be expressed by the relation $a^i = \sum_{j \neq i} \alpha_j a^j / \alpha_i$. We say that the first is a linear combination of the others. The vector 0 is by itself dependent. If a set of vectors is linearly independent then any subset is also independent.

2. A matrix all of whose entries are 0/1 which has exactly one 1 in each row and each column is called a *permutation* matrix. Each such matrix corresponds to a unique permutation of the integers $\{1, 2, ..., n\}$ if the matrix is of size $n \times n$. For example, the permutation $\{4, 5, 2, 3, 1\}$ of the first five integers would correspond to the 5×5 matrix P looks like:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   A permutation is said to be odd (even) depending on the oddness (evenness) of the number of reversals in the order in which the numbers appear in relation to the natural ordering of these numbers, for example $\{1, 2, 3, 4, 5\}$ in the above illustration. Thus, the above permutation is even.

3. Determinants are defined for square matrices only. The determinant of a matrix A of size $n \times n$ is given by the relation $|A| = \sum (\pm) a_{1,i_1} a_{2,i_2} ... \ a_{n,i_n}$ where $(i_1, i_2, ..., i_n)$ is a permutation of the integers $\{1,\ 2, ..., n\}$; the sum is over all permutations with $+1$ as the first part for even and -1 for the odd permutations. There are some well known properties of determinants that are listed below:

(a) Interchange of two columns (rows) changes only the sign of the determinant

(b) $|A^t| = |A|$

(c) If only one column (row) is multiplied by a scalar λ then the determinant is also scaled up by the same factor λ; this in turn implies that $|\lambda A| = \lambda^n |A|$; this also implies that if the matrix has a column (row) in which all entries are zero then the determinant is zero.

(d) If we denote by $a_{i,j}$ the entry of a matrix in the i^{th} row and j^{th} column and by $A_{i,j}$ the determinant of the submatrix obtained by the deletion of the i^{th} row and the j^{th} column from A multiplied by $(-1)^{i+j}$ then it is easy to show by grouping terms in the expression for the determinant that $|A| = \sum_{j=1}^{n} a_{i,j}.A_{i,j}$ and this is often referred to as the cofactor expansion and $A_{i,j}$ as the cofactor of $a_{i,j}$. The matrix A^+ whose elements $a_{i,j}^+ = A_{i,j}$ is called the adjoint of matrix A and it is easy to show that $AA^+ = A^+A = |A|I$; this implies if an entire column (row) of a square matrix is zero then its determinant is also zero.

(e) Adding a multiple of some other column (row) to a column (row) does not change the determinant; this together with the previous result imply that if the columns (rows) of a matrix are linearly dependent then the determinant of the matrix is zero; the converse follows from the above definition of A^+.

(f) If A and B are two square matrices of the same size then $|AB| = |A|.|B|$; similar result for rectangular matrices is called the Binet-Cauchy formula and is more complicated.

(g) Square matrices whose determinants are not equal to zero are called nonsingular matrices; all other matrices are singular. Given a square matrix A if $\exists$ a matrix $B \ni AB = BA = I$ then we say that B is the (multiplicative) inverse of A and denote it by A^{-1}. Clearly if $|A| \neq 0$ then the relation $AA^+ = A^+A = |A|I$ yields the relation $AB = BA = I$ if we let $B = A^+/|A|$ and hence this is the inverse in this case. If A^{-1} exists then since $I = AA^{-1}$, $1 = |I| = |A|.|A^{-1}|$ neither $|A|$ nor $|A^{-1}|$ can be zero as their product equals 1. Hence, if A^{-1} exists then $|A| \neq 0$. To show that the inverse is unique if it exists, let B and D both be inverse of a square matrix A. Hence $DAB = DI = D$; but $DAB = IB = B$; hence $D = B$ and hence the uniqueness. It is easy to show that $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^t)^{-1} = (A^{-1})^t$ assuming all these exist.

(h) The system of equations $Ax = b$ (the same as the system $\sum_j a_{i,j}x_j = b_i$ for $i = 1,\ 2, ...,\ m$) has a solution if the vector b is a linear combination of the column vectors of A. $Ax = 0$ has a nonzero solution if the column vectors A are linearly dependent. Thus, $Ax = b$ has a unique solution if the columns of A are linearly independent. If the rows of A are linearly dependent, then either the system has a redundant equation or is inconsistent (depending on whether the rows of the matrix $[A, b]$ are similarly dependent or not). If we assume that rows of A are linearly independent then uniqueness is equivalent to nonsingularity of A.