

0.1 Example of Simplex Method (Two Phases):

Consider the following example: $\min z : x \geq 0$ and

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 + 4x_5 &= z \\ 4x_1 + 2x_2 + 13x_3 + 3x_4 + x_5 &= 17 \\ x_1 + x_2 + 5x_3 + x_4 + x_5 &= 7 \end{aligned}$$

After introducing artificial variables and getting initial canonical form for Phase I the table looks like:

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
4	2	13	3	1	1	0	0	0	17
1	1	5	1	1	0	1	0	0	7
2	1	2	1	4	0	0	1	0	0
-5	-3	-18	-4	-2	0	0	0	1	-24

Basic variables are $\{v_1, v_2, -z, -w\}$. $\bar{c}_s = -18$ is shown in bold and is negative. So the variable selected to increase (enter the basis) is x_3 . The pivot element is also shown in bold and is $a_{1,3} = 13$. After one pivot the system looks like

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
$\frac{4}{13}$	$\frac{2}{13}$	1	$\frac{3}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	$\frac{17}{13}$
$-\frac{7}{13}$	$\frac{3}{13}$	0	$-\frac{2}{13}$	$\frac{8}{13}$	$-\frac{5}{13}$	1	0	0	$\frac{6}{13}$
$\frac{18}{13}$	$\frac{9}{13}$	0	$\frac{7}{13}$	$\frac{50}{13}$	$-\frac{2}{13}$	0	1	0	$-\frac{34}{13}$
$\frac{7}{13}$	$-\frac{3}{13}$	0	$\frac{2}{13}$	$-\frac{8}{13}$	$\frac{18}{13}$	0	0	1	$-\frac{6}{13}$

The next table looks like:

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
$\frac{3}{8}$	$\frac{1}{8}$	1	$\frac{2}{8}$	0	$\frac{1}{8}$	$-\frac{1}{8}$	0	0	$\frac{10}{8}$
$-\frac{7}{8}$	$\frac{3}{8}$	0	$-\frac{2}{8}$	1	$-\frac{5}{8}$	$\frac{13}{8}$	0	0	$\frac{6}{8}$
$\frac{38}{8}$	$-\frac{6}{8}$	0	$\frac{12}{8}$	0	$\frac{18}{8}$	$-\frac{50}{8}$	1	0	$-\frac{44}{8}$
0	0	0	0	0	1	1	0	1	0

Now if remove all traces of artificiality since we have reached the end of Phase I with $z = 0$, we get:

x_1	x_2	x_3	x_4	x_5	$-z$	RHS
$\frac{3}{8}$	$\frac{1}{8}$	1	$\frac{2}{8}$	0	0	$\frac{10}{8}$
$-\frac{7}{8}$	$\frac{3}{8}$	0	$-\frac{2}{8}$	1	0	$\frac{6}{8}$
$\frac{38}{8}$	$-\frac{6}{8}$	0	$\frac{12}{8}$	0	1	$-\frac{44}{8}$

Now we continue Phase II. The next table looks like:

x_1	x_2	x_3	x_4	x_5	$-z$	RHS
$\frac{2}{3}$	0	1	$\frac{1}{3}$	0	0	1
$-\frac{7}{3}$	1	0	$-\frac{2}{3}$	1	0	2
3	0	0	1	2	1	-4

At this point we have an optimal solution: $x_1 = x_4 = x_5 = 0; x_2 = 2; x_3 = 1; z = 4$.

1 Revised Simplex Method

This is also known as *the simplex method using multipliers*. Consider a linear program:

$$\begin{aligned} \min z : [Bx^B + A^{NB}x^{NB}] &= b \\ c^B x^B + c^{NB} x^{NB} - z &= 0 \end{aligned}$$

where B is current basis. Assume that the equations are linearly independent. Hence B^{-1} exists. Hence the inverse of $\begin{bmatrix} B & 0 \\ c^B & 1 \end{bmatrix}$ exists and is given by:

$$\begin{bmatrix} B^{-1} & 0 \\ -c^B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} B^{-1} & 0 \\ \pi & 1 \end{bmatrix}$$

The canonical form corresponding to this basis is obtained by multiplying the system (enlarged by including the z row) by $\begin{bmatrix} B^{-1} & 0 \\ -c^B B^{-1} & 1 \end{bmatrix}$. The last row of the result is $[\bar{c}, 1]$ and is obtained by multiplying the vector $[\pi, 1]$ times the original (enlarged) matrix. Using this we can check if $\bar{c} \geq 0$ or not. If it is not, we can also find s that satisfies $\bar{c}_s = \min \bar{c}_j < 0$. This yields the *entering variable*. To find the updated column (that in the canonical form) of this variable we need to multiply $\begin{bmatrix} B^{-1} & 0 \\ -c^B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A_{.s} \\ c_s \end{bmatrix}$. Similar process gives the updated RHS by the relation $\begin{bmatrix} B^{-1} & 0 \\ -c^B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} b \\ -z^0 \end{bmatrix}$. Now doing a pivot operation on the subsystem that includes these three yields the new inverse and the process is repeated until the usual termination conditions are observed. That the new inverse is obtained in this manner follows by observing that if an identity matrix is appended to the system we always have the inverse in its position.

One advantage of this procedure is that many entries in the updated form need not be calculated; this results in saving computations if the number of variables exceeds the number of equations by a significant amount. *While this advantage is stressed in many texts, what is more important is that we can formulate problems with enormous number of variables that arise from combinatorial explosion in some applications; these are, in my opinion, the main advantage of this method.* This often goes under the name of *(delayed) column generation technique*.

We now show the above example in Revised Simplex method:

The first table (original data) is still the same

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
4	2	13	3	1	1	0	0	0	17
1	1	5	1	1	0	1	0	0	7
2	1	2	1	4	0	0	1	0	0
-5	-3	-18	-4	-2	0	0	0	1	-24

However, we use for calculations only the following part:

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
		13			1	0	0	0	17
		5			0	1	0	0	7
		2			0	0	1	0	0
-5	-3	-18	-4	-2	0	0	0	1	-24

and when we perform the pivot operation we get:

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
					$\frac{1}{13}$	0	0	0	$\frac{17}{13}$
					$-\frac{5}{13}$	1	0	0	$\frac{6}{13}$
					$-\frac{2}{13}$	0	1	0	$-\frac{34}{13}$
					$\frac{18}{13}$	0	0	1	$-\frac{6}{13}$

Basis: $(x_1, v_2, -z, -w)$. \hat{B}^{-1} is shown under $(v_1, v_2, -z, -w)$ [this will be always the under these columns]. The vector $(-d_B B^{-1}, 1) = (\frac{18}{13}, 0, 0, 1)$.

The vector $\begin{bmatrix} b_1 \\ b_2 \\ -z_0 \\ -w_0 \end{bmatrix} = \begin{bmatrix} \frac{17}{13} \\ \frac{6}{13} \\ -\frac{34}{13} \\ -\frac{6}{13} \end{bmatrix}$. The vector $\bar{d} = d = d_B B^{-1} A = \begin{bmatrix} \frac{7}{13} & -\frac{3}{13} & 0 & \frac{2}{13} & -\frac{8}{13} & \frac{18}{13} & 0 & 0 & 1 \end{bmatrix}$.

Hence entering variable is x_5 and its updated column is given by $\hat{B}^{-1} A_{.5} =$

$\frac{1}{13}$	0	0	0	1	$\frac{1}{13}$
$-\frac{5}{13}$	1	0	0	1	$\frac{8}{13}$
$-\frac{2}{13}$	0	1	0	4	$\frac{50}{13}$
$\frac{18}{13}$	0	0	1	-2	$-\frac{8}{13}$

Thus, we have our new skeletal system

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
				$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	$\frac{17}{13}$
				$\frac{8}{13}$	$-\frac{5}{13}$	1	0	0	$\frac{6}{13}$
				$\frac{50}{13}$	$-\frac{2}{13}$	0	1	0	$-\frac{34}{13}$
				$-\frac{8}{13}$	$\frac{18}{13}$	0	0	1	$-\frac{6}{13}$

Ratio test yields the bold element as the new pivot element and we do the pivot on the skeletal system. The new basis is $(x_1, x_5, -z, -w)$ and we know Phase I

has come to an end. The new table looks like

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	$-w$	RHS
					$\frac{1}{8}$	$-\frac{1}{8}$	0	0	$\frac{10}{8}$
					$-\frac{5}{8}$	$\frac{13}{8}$	0	0	$\frac{6}{8}$
					$\frac{18}{8}$	$-\frac{50}{8}$	1	0	$-\frac{44}{8}$
					1	1	0	1	0

At this point we remove the last row and the column under $-w$ to get a skeletal system for the original problem. This yields:

x_1	x_2	x_3	x_4	x_5	v_1	v_2	$-z$	RHS
					$\frac{1}{8}$	$-\frac{1}{8}$	0	$\frac{10}{8}$
					$-\frac{5}{8}$	$\frac{13}{8}$	0	$\frac{6}{8}$
					$\frac{18}{8}$	$-\frac{50}{8}$	1	$-\frac{44}{8}$

with the basis $(x_3, x_5, -z)$. $\bar{c} = c - c_B B^{-1} A = \boxed{\frac{38}{8} \quad -\frac{6}{8} \quad 0 \quad \frac{12}{8} \quad 0 \quad 1}$. Thus entering variable at this stage (of Phase II) is x_2 . To find the variable that drops out we need the new skeletal structure which is obtained by "erecting" the column for x_2 in the updated table and this looks like:

x_1	x_2	x_3	x_4	x_5	$-z$	RHS
	$\frac{1}{8}$		$\frac{2}{8}$	0	0	$\frac{10}{8}$
	$\frac{13}{8}$		$-\frac{2}{8}$	1	0	$\frac{6}{8}$
	$-\frac{6}{8}$		$\frac{12}{8}$	0	1	$-\frac{44}{8}$

The bold element in the second row is the new pivot element and performing this pivot operation gives us the table:

x_1	x_2	x_3	x_4	x_5	$-z$	RHS
			$\frac{1}{3}$	0	0	1
			$-\frac{2}{3}$	1	0	2
			1	2	1	-4

with basis $(x_3, x_2, -z)$. Calculating the new $\bar{c} = c - c_B B^{-1} A = \boxed{3 \quad 0 \quad 0 \quad 1 \quad 2 \quad 1}$. We now have reached optimality since $\bar{c} \geq 0$.

Optimal solution is $x_3 = 1; x_2 = 2; -z = -4; x_3 = x_4 = x_5 = 0$.

Instead of storing the entire inverse, we can store the inverse between successive steps. These are elementary matrices and hence easy to store. This is discussed in your book under *product form of the inverse*.

Example II:

$$\begin{aligned} \min & 40x_1 + 36x_2 \\ & x_1 \leq 8 \\ & x_2 \leq 10 \\ & 5x_1 + 3x_2 \geq 45 \\ & x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

Initial canonical form for Phase I looks like:

x_1	x_2	s_1	s_2	s_3	v_1	$-z$	$-w$
1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
5	3	0	0	-1	1	0	0
40	36	0	0	0	0	1	0
-5	-3	0	0	1	0	0	1

RHS					
8	1	0	0	0	0
10	0	1	0	0	0
45	0	0	1	0	0
0	0	0	0	1	0
-45	0	0	0	0	1

The skeletal system that the revised simplex method works with is as shown below: first we get $\begin{bmatrix} -5 & -3 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ from which we conclude that the entering variable is x_1 . Now we get the column (updated) for this variable and the RHS as shown below and affix an identity matrix next to RHS.

x_1	x_2	s_1	s_2	s_3	v_1	$-z$	$-w$
1							
0							
5							
40							
-5	-3	0	0	1	0	0	1

RHS					
8	1	0	0	0	0
10	0	1	0	0	0
45	0	0	1	0	0
0	0	0	0	1	0
-45	0	0	0	0	1

Pivoting on this system we get:

RHS					
8	1	0	0	0	0
10	0	1	0	0	0
5	-5	0	1	0	0
-320	-40	0	0	1	0
-5	5	0	0	0	1

The matrix next to the (updated RHS) is the new inverse. $\begin{bmatrix} 5 & 0 & 0 & 0 & 1 \end{bmatrix}$ corresponds to the new simplex multipliers $[(\sigma, 1)]$ (σ is used for Phase I and π for Phase II).

5	0	0	0	1
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1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
5	3	0	0	-1	1	0	0
40	36	0	0	0	0	1	0
-5	-3	0	0	1	0	0	1

0	-3	5	0	1	0	0	1
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which is the bottom row of the updated matrix. This lets us know that the

entering variable is x_2 . So we construct the new skeletal system by

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline -5 & 0 & 1 & 0 & 0 \\ \hline -40 & 0 & 0 & 1 & 0 \\ \hline 5 & 0 & 0 & 0 & 1 \\ \hline \end{array}
 \times
 \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline 3 \\ \hline 36 \\ \hline -3 \\ \hline \end{array}
 =
 \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \mathbf{3} \\ \hline 36 \\ \hline -3 \\ \hline \end{array}$$

to get

$$\begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \mathbf{3} \\ \hline 36 \\ \hline -3 \\ \hline \end{array}
 \begin{array}{|c|c|c|c|c|c|} \hline 8 & 1 & 0 & 0 & 0 & 0 \\ \hline 10 & 0 & 1 & 0 & 0 & 0 \\ \hline 5 & -5 & 0 & 1 & 0 & 0 \\ \hline -320 & -40 & 0 & 0 & 1 & 0 \\ \hline -5 & 5 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

and pivot on this system to get the new inverse and the new RHS. This process is repeated till optimality or unboundedness. I have shown Phase I so far. When Phase I is over, we remove all artificial variables, and the last row and column of the inverse and continue.