1 Introduction

Matroids are combinatorial structures that abstract the properties of linear independence of vector spaces and related properties of graphs. One of the first papers is due to **H.Whitney** (1935). There is a considerable amount of work since then and this is a rich area dealing with structural as well optimization results. Occasionally there is a combination of both as in testing of total unimodularity. There are several aspects to matroids and each can be used to define them. We will show this and the relationship between these by considering them one by one.

2 General Definitions

2.1 Independence

This is a direct generalization of the notion of linear independence in vector spaces.

Definition 1 (IS) *Let $\mathcal{F}$ be a nonempty family of subsets of a finite set E that satisfies the property: $F \in \mathcal{F}$, $G \subset F \Longrightarrow G \in \mathcal{F}$. Then, the pair $[E, \mathcal{F}]$ is called an* independence system*. Members of $\mathcal{F}$ are called* independent sets*.*

Definition 2 (MI.1) *An independence system $[E, \mathcal{F}]$ satisfying the property:*

$$[F \in \mathcal{F}, G \in \mathcal{F}, |G| > |F|] \Longrightarrow \exists\, g \in G - F \ni F \cup \{g\} \in \mathcal{F}$$

is called a matroid.

Definition 3 (MI.2) *An independence system $[E, \mathcal{F}]$ satisfying the property: [All maximal independent subsets of a set $A \subset E$, have the same size.] is called a* matroid. *The size of such a maximal subset is called the rank $r(A)$ of A.*

Definition 4 (MR.1) *Given a finite set E, a function $r : 2^E \rightarrow Z_+$ is called a* rank function *if it satisfies:*

(i)$r(\phi) = 0$;
(ii)$S \subseteq T \subseteq E \Longrightarrow r(T) \geq r(S)$;
(iii)$r(S) \leq |S|$;
(iv)$r(S \cup T) + r(S \cap T) \leq r(S) + r(T) \forall S, T \subseteq E$
then the pair $[E, r]$ is called a *matroid.*

Definition 5 (MR.2) *Given a finite set E, a function $r : 2^E \rightarrow Z_+$ is called a* rank function *if it satisfies:*

(i)$r(\phi) = 0$
(ii)$S \subseteq T \subseteq E \Longrightarrow r(T) \geq r(S)$;
(iii)$r(S \cup \{e\}) = r(S)$ or $r(S) + 1 \; \forall \; S \subset E$.
(iv)$[r(S \cup \{e_1\}) = r(S \cup \{e_2\}) = r(S)] \Longrightarrow r(S \cup \{e_1, e_2\}) = r(S)$
then the pair $[E, r]$ is called a *matroid.*

Definition 6 (MC.1) Circuits *form a clutter (a family of subsets none of which is contained in another) on E. If $\mathcal{C}$ is a clutter on E that satisfies the property:*

$$C_1 \in \mathcal{C}; C_2 \in \mathcal{C}; e_1 \in C_1 \cap C_2; e_2 \in C_1 - C_2$$

$$\Downarrow$$

$$\exists \, C_3 \in \mathcal{C}; C_3 \subset C_1 \cup C_2; e_2 \in C_3; e_1 \notin C_3$$

then the pair $[E, \mathcal{C}]$ is called a matroid.

Definition 7 (MC.2) *If $\mathcal{C}$ is a clutter on E that satisfies the property:*

$$C_1 \in \mathcal{C}; C_2 \in \mathcal{C}; e_1 \in C_1 \cap C_2$$

$$\Downarrow$$

$$\exists \, C_3 \in \mathcal{C}; C_3 \subseteq C_1 \cup C_2 - \{e_1\}$$

then the pair $[E, \mathcal{C}]$ is called a matroid.

Definition 8 (MB.1) *If $\mathcal{B}$ is a clutter on E that satisfies the property:*

$$[B \in \mathcal{B}; B' \in \mathcal{B}; e' \in B' - B] \Longrightarrow [\exists \, e \in B - B' \ni \{B' - e' + e\} \in \mathcal{B}]$$

then the pair $[E, \mathcal{B}]$ is called a matroid.

Definition 9 (MB.2) *If $\mathcal{B}$ is a clutter on E that satisfies the property:*

$$[B \in \mathcal{B}; B' \in \mathcal{B}; e' \in B' - B] \Longrightarrow [\exists \, e \in B - B' \ni \{B - e + e'\} \in \mathcal{B}]$$

then the pair $[E, \mathcal{B}]$ is called a matroid.

There are many other ways of defining matroids some of which are quite useful. First we show how to relate these concepts.

Given **??**: (i)$r(S) = \max_{T \subseteq S; T \in \mathcal{F}} |T|$; (ii) $\mathcal{C}$ is the collection of *minimal* (*in the set theoretic sense*) subsets C of $E \ni C \notin \mathcal{F}$; (iii)$\mathcal{B}$ is the collection of *maximal* (*again in the set theoretic sense*) subsets B of $E \ni B \in \mathcal{F}$.

Given **??**: (i)$\mathcal{F}$ is the collection of subsets F of $E \ni |F| = r(F)$; (ii) $\mathcal{B}$ and $\mathcal{C}$ are defined as above in **??**.

Given **??**:(i)$\mathcal{F}$ is the collection of subsets F of $E \ni \exists$ no $C \in \mathcal{C}; C \subseteq F$; (ii) $\mathcal{B}$ and r are defined as in **??**.

Given **??**: (i)$\mathcal{F}$ is the collection of subsets F of E that are subsets of some member of $\mathcal{B}$; (ii)r and $\mathcal{C}$ are defined as in **??**.

Exercise: Show the equivalence of all these definitions.

3 Examples

1. E is a (finite) set of vectors in a vector space and F is the collection of linearly independent vectors. This type of matroids are called *linear* (or *representable*) matroids.

2. E is the set of edges of an undirected graph and F is the collection os subsets of edges that form no loop (these are called *forests*). This is really an example of the previous type. This type is known as *graphic* matroid

3. E is a finite set and $\mathcal{F} = \{S : S \subseteq E; |S| \leq k\}$. Such matroids are called *cardinality* matroids. If $k = |E|$, the matroid is called a *free* matroid. If $E = \cup E_i$ where E_i are disjoint, and $\mathcal{F} = \{F : F = \cup F_i; |F_i| \leq k_i; F_i \subseteq E_i\}$, then the matroid is called a *partition* matroid.

4. E is the set of arcs in a directed graph and F is the collection of subsets F of E with the property that no more than one arc enters (leaves) a node.

5. As in example 2 except we have any one additional edge. This is called a 1-*forest*.

6. Making matroids from other matroids: Let $M = [E, \mathcal{F}] = [E, \mathcal{B}] = [E, \mathcal{C}] = [E, r]$ be a matroid. Then:

(i) Let $\mathcal{F}_k = [F : F \in \mathcal{F}; |F| \leq k]$. Then, $M_k = [E, \mathcal{F}_k]$ is a matroid called the *k-truncation* of M.

(ii) Let $\not{\mathcal{B}} = [\bar{B} : \bar{B} = E - B$ for some $B \in \mathcal{B}]$. Then, $M^* = [E, \not{\mathcal{B}}]$ is a matroid called the *dual* of M. *Please note that*: $(M^*)^* = M$.

(iii) Let $\mathcal{F} \backslash e = [F : F \in \mathcal{F}; e \notin F]$. Then $M \backslash e = [E - e, \mathcal{F} \backslash e]$ is a matroid *obtained by* deleting e *in the original matroid* M.

(iv)$M/e = [E - e, \mathcal{F}/e] = (M^* \backslash e)^*$ is a matroid *obtained by* contracting e *in the original matroid. Matroids obtained by a sequence of deletions and contractions* (*the order does not matter*) *are called* minors *of the original matroid.*

Some boring but useful results:

1. $r(S \cup T) \leq r(S) + r(T)$

2. $r(G \cup \{e\}) = r(G) \; \forall \; e \in F \Longrightarrow r(G \cup F) = r(G)$; maximal set $H \supseteq G \ni r(H) = r(G)$ is called the *span* of G. This definition is similar to the one in vector spaces.

3. $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$. A function satisfying this property is said to be a *submodular* function.

4. $r(S) \leq |S|$.

5. $[e \notin S, \exists \, C \in C, e \in C \subseteq S \cup \{e\}] \Longleftrightarrow [r(S \cup \{e\}) = r(S)]$.

6. $[B \in \mathcal{B}, B' \in \mathcal{B}] \Longrightarrow [|B| = |B'|]$. This value is called the *rank* of the matroid.

7. Single element dependent sets are called *self loops*. Dependent sets whose cardinality is 2 are called *parallel elements*. Two elements that are in parallel in M^* are said to be in *series* in M.

8. A set S is said to be *closed (flat, subspace)* if $r(S \cup e) = r(S) + 1 \; \forall \; e \notin S$.

9. A monotone nonnegative integral submodular function f that also satisfies the relation $f(S) \leq |S|$ is the rank function of a matroid.

10. Let L be a *lattice* of subsets of a finite set $E \ni$(i)it is ordered by inclusion; (ii)closed under intersection ; and (iii)contains ϕ, and S. Then, $A \cup B \subseteq A \vee B; A \wedge B = A \cap B$. Let $f : L \longmapsto \mathbf{Z}_+$ be a nonnegative integral submodular function with $f(\phi) = 0$; i.e. $f(A \wedge B) + f(A \vee B) \leq f(A) + f(B) \; \forall \; A, B$. Then $\rho(S) = \inf_{T \in L}[f(T) + |S - T|]$ is the rank function of a matroid whose $\mathcal{F}$ is given by: $\mathcal{F} = [S : S \subset E; f(T) \geq |S \cap T| \; \forall \; T \in L]$.

11. Let $[E, \mathcal{F}]$ be an independence system satisfying the property: $\forall \; S \subset E$, every maximal subset T of S which is a member of $\mathcal{F}$ has the same cardinality (and we call this value the rank). Then $[E, \mathcal{F}]$ is a matroid.

12. Let M/S be obtained by contraction of $S \subset E$. Circuits of this matroid are the minimal members of $C_S = [C_S : C_S = C \cap S \neq \phi$ for some $C \in C]$. Independent sets of this matroid are given by $F_S = [J : J \subset S; J \cup K \in F$ for some maximal independent set K in $E - S]$.

Theorem 1 *Let $\mathcal{C}$ be a clutter on E satisfying*: $[C_1 \in \mathcal{C}, C_2 \in \mathcal{C}, e \in C_1 \cap C_2] \Longrightarrow [\exists \; C_3 \in \mathcal{C}, C_3 \subseteq C_1 \cup C_2 - \{e\}]$. *Then* $[E, \mathcal{C}]$ *is a matroid.*

Proof: We need to show that $[C_1 \in \mathcal{C}, C_2 \in \mathcal{C}, e_1 \in C_1 \cap C_2$, and $e_2 \in C_1 - C_2] \Longrightarrow [\exists \; C_3 \in \mathcal{C}, C_3 \subseteq C_1 \cup C_2, e_2 \in C_3, e_1 \notin C_3]$.

Suppose the theorem is false for some set $\{e_1, e_2, C_1, C_2\}$. For this specific e_2, choose the remaining triple so that among all violations of the theorem with $e_2, C_1 \cup C_2$ is minimal. By hypothesis, $\exists \; C_3 \subset C_1 \cup C_2 - e_1$. Since $\mathcal{C}$ is a clutter, ($C_3 \subset C_1$ is false and hence) $\exists \; e_3 \in (C_2 - C_1) \cap C_3$. Again by hypothesis, $\exists \; C_4 \subset C_3 \cup C_2 - e_3$. Since $C_4 \subset C_2$ is also false, $\exists \; e_4 \in (C_1 - C_2) \cap C_4$. $e_2 \in C_4 \Longrightarrow e_2 \in C_3$ a contradiction to the hypothesis that the set $\{e_1, e_2, C_1, C_2\}$ is a violator of the theorem. Hence $e_2 \in C_1 - C_4$. But $e_4 \in C_1 \cap C_4$. $e_3 \in C_1 \cup C_2; e_3 \notin C_1; e_3 \notin C_4$; hence $e_3 \notin C_1 \cap C_4$. Hence $C_1 \cup C_4 \subset C_1 \cup C_2$. $e_2 \in C_1 - C_4; e_4 \in C_1 \cap C_4; C_1 \cup C_4 \subset C_1 \cup C_2$. By minimality of the violation of $C_1 \cup C_2, \exists \; C \in \mathcal{C} \ni e_2 \in C; C \subseteq C_1 \cup C_4 - e_4$. But since $e_2 \in C$; $C \subset C_1 \cup C_2$ we have $e_1 \in C$. (otherwise C does the job of the theorem). Thus the theorem is violated for e_1, e_2, C and C_2. But $e_4 \notin C; e_4 \notin C_2; e_4 \in C_1$. Hence $C \cup C_2 \subset C_1 \cup C_2$. This again contradicts the minimality of the violator chosen.□

4 (Single) Matroid (Linear) Optimization Problems

Given a matroid $M = [E, \mathcal{F}] = [E, \mathcal{B}]$ and $c : E \longmapsto \mathbf{R}$:

(i)max $c(F) : F \in \mathcal{F}$ where $c(F) = \sum_{e \in F} c_e$.

(ii)max $c(B) : B \in \mathcal{B}$ where $c(B) = \sum_{e \in B} c_e$.

(iii)$\max_{F \in \mathcal{F}} \min_{e \in F} c_e$ OR (iii') $\min_{F \in \mathcal{F}} \max_{e \in F} c_e$

(iv)$\max_{B \in \mathcal{B}} \min_{e \in B} c_e$ OR (iii') $\min_{B \in \mathcal{B}} \max_{e \in B} c_e$

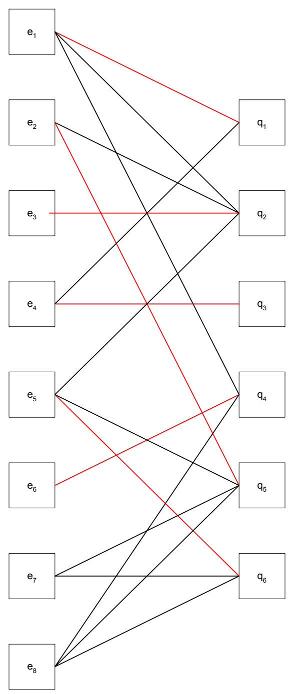
(v)Given $B \in \mathcal{B}$, let c_B be the ordered vector (in decreasing order) of c_e of e in B.

Find $B \in \mathcal{B} \ni c_B \geq c_{B'}\ \forall\ B' \in \mathcal{B}$.

4.1 Transversal Matroids and Gale Problem

Let $E = [e_1, e_2, ..., e_n]$ be a finite set and let $Q = [q_1, q_2, ..., q_m]$ be a family of (*not necessarily distinct*) subsets of E. Then the set $T = [e_{j_1}, e_{j_2}, ..., e_{j_t}]$ is called a *partial transversal* of size t of Q if the elements in T are distinct members of E and there are distinct indices i_k, $\ni e_{j_k} \in q_{i_k}$ for $1 \leq k \leq t$. The set T is called a *transversal or a system of distinct representatives* if $t = m$. We can visualize a partial transversal by the representation: $\begin{matrix} e_{j_1} & e_{j_2} & . & . & e_{j_t} \\ q_{i_1} & q_{i_2} & . & . & q_{i_t} \end{matrix}$.
We can visualize this from the point of view of matching on bipartite graphs as follows: One set of nodes correspond to the set E and the other to the family

Q as shown



A matching in this graph corresponds to a partial transversal. In the above diagram, $\{e_1, e_2, ..., e_6\}$ constitute a partial transversal.

Theorem 2 *Let E and Q be as above. Then $M_a = [E, \mathcal{F}]$ is a matroid if $\mathcal{F}$ is a family of partial transversals of Q. $M_b = [Q, \mathcal{Q}]$ is a matroid if $\mathcal{Q}$ is the collection of subsets of Q that have transversals.*

Proof: It should be clear that both are independence systems. We need to show the additional axiom for matroid. Suppose we have two members F and G of F with $|G| = |F| + 1$. Then their representations look like:

$$F = \begin{array}{ccccc} e_{j_1} & e_{j_2} & . & . & e_{j_{t-1}} \\ q_{i_1} & q_{i_2} & . & . & q_{i_{t-1}} \end{array} ; G = \begin{array}{ccccc} e_{j'_1} & e_{j'_2} & . & . & e_{j'_t} \\ q_{i'_1} & q_{i'_2} & . & . & q_{i'_t} \end{array}$$

Suppose $e_{j_k} \equiv e_{j'_k}$ and $q_{i_k} \equiv q_{i'_k}$ for $1 \leq k \leq r$. Since $|G| > |F|$, $\exists\ i'_k \ni$ this i' is distinct from all those in F. If $\begin{array}{c} e_{j_k} \\ q_{i'_k} \end{array}$ is not in F we are done. On the other hand suppose that $e_{j'_k} = e_{j_u}$ for some $u \geq r + 1$ then replacing q_{i_u} by $q_{i'_k}$ yields another representation for F which shares one more in common with G. Repeating this process we will eventually be able to grow F by an element from G. This shows that M_a is a matroid. To show that M_b is a matroid use a similar argument.

This theorem can also be proved using matching theory on the above graph.

4.2 Optimal Assignments in an Ordered set

Lemma 3 *Let $M = [E, \mathcal{F}]$ be a matroid. Let the elements of E be totally ordered and let this ordering be $e_1 > e_2 > ... > e_n$. $\exists$ an ordered subset $\{f_1, f_2, ..., f_k\} = F \in \mathcal{F}$ such that for any other ordered set $\{g_1, g_2, ..., g_l\} = G \in \mathcal{F}$, we have $l \leq k$ and $f_i \geq g_i$ for $i = 1, 2, ..., l$. We call this set F a Gale-optimal set.*

Proof Perform the following algorithm known as the "Greedy Algorithm" on the above data.

Greedy Algorithm:

1. $F \longleftarrow \phi$
2. **for** $i \longleftarrow 1$ **to** n
3.   **if** $F \cup \{e_i\} \in \mathcal{F}$
4.   **then** $F \longleftarrow F \cup \{e_i\}$
5. **return** F

Claim The set F produced by the above algorithm is Gale-optimal.

Proof of Claim Suppose not. Then there exists a $G \in \mathcal{F}$ such that the conditions claimed in the lemma do not hold. Let both F and G be ordered as per the relation on E. Let $k = \min[j : g_j > f_j]$. Just before the greedy algorithm selected the element f_j, the subset of elements from F already selected is $\hat{F} = \{f_i : i < j\} \in \mathcal{F}$. But $\hat{G} = \{g_i : i \leq j\} \in \mathcal{F}$. $\left|\hat{G}\right| > \left|\hat{F}\right|$ and since $g_j > f_j$ and $g_i > g_j$ for all $i < j$, each element of $\hat{G}$ is "greater" than f_j. Moreover, since M is a matroid, $\exists g \in \hat{G} - \hat{F}$ such that $\hat{F} \cup \{g\} \in \mathcal{F}$. This contradicts the fact that F was selected by the greedy algorithm. This also completes the proof of the lemma.

Lemma 4 *Let $M = [E, F]$ be an independence system for which the greedy algorithm solves the problem: $\max_{F \in \mathcal{F}} \sum_{e \in F} w(e)$ for all w. Then, M is a matroid.*

Proof Let F, G be two elements in $\mathcal{F}$ and let $|G| > |F| = k$. Choose w as follows:

$$w(e) = \begin{cases} k+2 & e \in F \\ k+1 & e \in G - F \\ 0 & e \notin G \cup F \end{cases}$$

The greedy algorithm will first select all elements of F. But since

$$\begin{aligned}\sum_{e\in G} w(e) &= (k+2)(|G\cap F| + (k+1)\,|G-F| \\ &\geq (k+1)^2 \\ &> (k+2)k \\ &= \sum_{e\in F} w(e)\end{aligned}$$

Hence the algorithm must select at least one element $g \in (G-F)$ such that $F \cup \{g\} \in \mathcal{F}$. This shows that M is a matroid.

4.2.1 Matroid Union & Partition

Theorem 5 (Nash-Williams) *Let $M = [E, \mathcal{F}]$ be a matroid. Let $\hat{E}$ be a set and $h : E \to \hat{E}$ be a mapping from E to $\hat{E}$. Let $h(F) = \cup_{e\in F} h(e)$. Let $\hat{\mathcal{F}} = \{\hat{F} : \hat{F} = h(F)$ for some $F \in \mathcal{F}\}$. Then $\hat{M} = [\hat{E}, \hat{\mathcal{F}}]$ is a matroid.*

Proof Let $\hat{F} = h(F); \hat{G} = h(G); \left|\hat{G}\right| > \left|\hat{F}\right|; \hat{F}, \hat{G} \in \hat{\mathcal{F}}$. Select F, G so that $\left|\hat{F}\right| = |F|$ and $\left|\hat{G}\right| = |G|$. This can be done since $\hat{e} \neq \hat{f} \implies h^{-1}(\hat{e}) \neq h^{-1}(\hat{f})$. Hence $|G| > |F|$. Hence $\exists\; p \in G - F$ such that $F \cup \{p\} \in \mathcal{F}$. Let $A = F \cap G; B_F, B_G$ be such that $|B_F| = |B_G|$ and for each $f \in B_F$, $\exists\, g \in B_G$ with $h(f) = h(g)$. For each element q in $F - A - B_F$ and each element r in $G - A - B_G$, we have $h(q) \neq h(r)$. If $g \in G - A - B_G$, we are done. If $g \in B_G$, replace $f \in F$ with $h(f) = h(g)$ by g and this increases the set A. Please note that $h(F), h(G)$ are not altered by this. After a finite number of steps, we get a $g \in G - A - B_G$ and hence we are done.

Corollary 6 *Let $M_i = [E, F_i]; 1 \leq i \leq k$ be matroids. Let $M = [E, \mathcal{F}^{(1)}]$ where $\mathcal{F}^{(1)} = \{F : F = \cup F_i$ for some collection $F_i \in \mathcal{F}_i\}$. Then M is a matroid and this operation is called the union of matroids and we write $M = \cup M_i$.*

Proof Let $E_i; 1 \leq i \leq k$ be k copies of E. These are considered distinct sets for now. Let $M^{(2)} = [\cup E_i, \mathcal{F}^{(2)}]$ where $\mathcal{F}^{(2)} = \{F : F = \cup F_i$ (with multiple copies of an element if it occurs many times) for some collection $F_i \in \mathcal{F}_i\}$. The difference between a member of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ is that the second may contains duplicate copies of an element while the first does not. It is very easy to show that $M^{(2)}$ is a matroid (sometimes referred to as disjoint union). Now let $h : \cup E_i \to E$ be the mapping that maps all copies of an element to the one copy in E. It is easy to see that $h(\mathcal{F}^{(2)}) = \mathcal{F}^{(1)}$. So the result follows from the above theorem.

But the above result does not provide for an oracle for the union given the oracles for individual matroids. This was done by J. Edmonds through an algorithm for "Matroid Partition". This algorithm produces the sets F_i given an $F \in \mathcal{F}^{(1)}$ or shows that the given $F \notin \mathcal{F}^{(1)}$. We describe this work below:

Theorem 7 *Let $M_i; 1 \leq i \leq k$ and M be as above. A set F can be partitioned into sets $F_i \in \mathcal{F}_i$ (i.e. $F \in \mathcal{F}^{(1)}$)if and only if*

$$|A| \leq \sum_{i=1}^{k} r_i(A) \qquad \forall A \subseteq F$$

Proof It is easy to show that this is a necessary condition. To show that it is sufficient, we use an algorithm.

Definition 10 *Let $F \subseteq A \subseteq E; F \in \mathcal{F}$ where $M = [E, \mathcal{F}]$ is a matroid. The set $S \subseteq A$ defined by the relation*

$$S = F \cup \{e \in A : F \cup \{e\} \notin \mathcal{F}\}$$

is called the span of F in A with respect to matroid M and is denoted by $T(F, A, M)$. Informally, it is the set F together with all elements in A that are dependent on F. This set can also be defined by the relation: It is the unique maximal (with respect to set inclusion) set satisfying the relations: (i)$F \subseteq S \subseteq A$ and (ii)$r(S) = r(F) = |F|$.

The algorithm uses the following "primitive operation": For an index i, a $F \in F_i$, and an element $e \in E - F$, check if $F \cup \{e\} \in F_i$ and if the answer is no find a circuit $C \subseteq F \cup \{e\}$ in matroid M_i.

4.2.2 Algorithm

Without loss, we assume that the set we want to partition is the set E itself. Start with any family $\{F_i\}; i = 1, 2, ..., n$ of disjoint sets satisfying the relations $F_i \in \mathcal{F}_i$. Any number of these may be empty. Let $H = \cup F_i$. We say that H is partitionable with respect to $\{M_i\}$. Let $e \in E - H$. We show how to find a set $A \subseteq H \cup \{e\}$ satisfying the condition

$$|A| > \sum_{i=1}^{k} r_i(A)$$

in which case we can not partition $H \cup \{e\}$ or show how to partition $H \cup \{e\}$.

Phase I:

Start with $j = 1; S_0 = E$. If there is an index $i(j)$ such that $\left|F_{i(j)} \cap S_{j-1}\right| < r_{i(j)}(S_{j-1})$, set $S_j = T(F_{i(j)}, S_{j-1}, M_{i(j)})$ and increase j by 1. $S_j \subsetneqq S_{j-1}$since

$$r_{i(j)}(S_j) = \left|F_{i(j)} \cap S_{j-1}\right| < r_{i(j)}(S_{j-1})$$

If such an index does not exist, we have come to the end of Phase I and we denote the last set by S_n. Clearly we have:

$$|F_i \cap S_n| = r_i(S_n) \qquad \forall i$$

Since $\{F_i\}$ are disjoint, this implies

$$|H \cap S_n| = \sum_{i=1}^{k} r_i(S_n)$$

Now if $e \in S_n - H$, we have

$$\begin{aligned} A &= (H \cap S_n) \cup \{e\} \subseteq S_n \\ |A| &= |H \cap S_n| + 1 \\ &> \sum_{i=1}^{k} r_i(S_n) \\ &\geq \sum_{i=1}^{k} r_i(A) \end{aligned}$$

indicating that the set $H \cup \{e\}$ is not partitionable.

On the other hand if $e \in E - (H \cup S_n)$, then we can partition $H \cup \{e\}$ and this is shown below:

Phase II:

Since the sets S_j are nested and since $e \notin S_n$ but $e \in S_0 = E$, there is an index h such that $e \notin S_h$ but $e \in S_j$ for $0 \leq j < h$.

If $F_{i(h)} \cup \{e\} \in \mathcal{F}_{i(h)}$, then set $F_{i(h)} \longleftarrow F_{i(h)} \cup \{e\}$ and we are done.

If not, there is a circuit $C \subseteq F_{i(h)} \cup \{e\}$ in matroid $M_{i(h)}$.

Claim: $C \nsubseteq S_{h-1}$

Proof: Since $[S_h = T(F_{i(h)}, S_{h-1}, M_{i(h)})]$, if $C \subseteq S_{h-1}$, then $e \in C$ would imply that $e \in S_h$. This contradicts our assumption that $e \notin S_h$.

Let m be the smallest index such that $C \nsubseteq S_m$. Note that $0 < m < h$.

Let $e' \in C - S_m$. Then $\{F_{i(h)} - \{e'\}\} \cup \{e\} \in \mathcal{F}_{i(h)}$. Replacing $F_{i(h)}$ by

$$\begin{aligned} F'_{i(h)} &= \{F_{i(h)} - \{e'\}\} \cup \{e\} \\ F'_i &= F_i; i \neq i(h) \end{aligned}$$

we have taken care of the element e but now have to take care of the element e'. We know that $e' \notin S_m$ and that $e' \in S_j; 0 \leq j < m$.

The most important observation here is the following: If we did Phase I with $\{H - e'\} \cup \{e\}$ instead of H, the sequence $[(F'_{i(1)}, S_1), ..., (F'_{i(m)}, S_m)]$ is the same as the sequence $[(F_{i(1)}, S_1), ..., (F_{i(m)}, S_m)]$. Since $m < h$, this process must terminate in less than h steps at which time we can include the element that is in place of e at that step.

This completes the description of the algorithm and the proof of the theorem. Using this algorithm we can also prove the following corollary to Nash-Williams theorem.

Corollary 8 *Let $M_i = [E, \mathcal{F}_i]; 1 \leq i \leq k$ be matroids. Let $M = [E, \mathcal{F}]$ where $\mathcal{F}^{(1)} = \{F : F = \cup F_i$ for some collection $F_i \in \mathcal{F}_i\}$. Then M is a matroid and this operation is called the union of matroids and we write $M = \cup M_i$.*

Proof We will use the definition MI.2 to do this. It is easy to verify that $[E, \mathcal{F}]$ is an independence system. Let H be any member of $\mathcal{F}$. Starting with this H by doing a Phase I of the algorithm we get a S_n such that

$$|H \cap S_n| = \sum_{i=1}^{k} r_i(S_n)$$

Moreover, we showed, in the above discussion, that for any $e \in S_n - H$, $H \cup \{e\}$ is not partitionable. Thus H is a maximal partitionable subset of $H \cup S_n$. To show that it is a maximum cardinality partitionable subset of $H \cup S_n$, let H' be any other partitionable subset of $H \cup S_n$. Since$H' \subseteq H \cup S_n$, it follows that

$$|H' - S_n| \leq |H - S_n|$$

Since H' is partitionable, from the "only if" part of the theorem above, we have

$$\begin{aligned} |H' \cap S_n| &\leq \sum_{i=1}^{k} r_i(|H' \cap S_n|) \\ &\leq \sum_{i=1}^{k} r_i(S_n) \\ &= |H \cap S_n| \end{aligned}$$

Adding the two we get the desired result that $|H'| \leq |H|$. Now let A be any subset of E. Suppose that the above H is any maximal partitionable subset of A. Phase II of the algorithm shows that for any $e \in E - (H \cup S_n)$, $H \cup \{e\}$ is partitionable. Since we assumed that H is a maximal partitionable subset of A, it follows that $A \subseteq H \cup S_n$. Since H is a maximum cardinality partitionable subset of $H \cup S_n$, it is also a maximum cardinality partitionable subset of A. Hence M is a matroid. This also gives the method of constructing "the oracle' for M given such for each M_i.

4.3 Matroid Intersection:

We now take up another optimization problem in matroids.

Problem Given two matroids $M_1 = [E, \mathcal{F}_1]$ and $M_2 = [E, \mathcal{F}_2]$ on the same ground set E and weights $w(e)$ for each element $e \in E$, we want to solve the problem

$$\max_{F \in \mathcal{F}_1 \cap \mathcal{F}_2} [\sum_{e \in F} w(e)$$

If $w(e) \equiv 1$ for all $e \in E$, this problem is known as cardinality version of the intersection problem. This version can be solved by using the partition algorithm>

Suppose $F \in \mathcal{F}_1 \cap \mathcal{F}_2$. Then $E - F$ contains a basis B_2^* of M_2^* the dual of M_2. Hence the set $F \cup B_2^*$ is partitionable with respect to matroids M_1 and M_2^*. Conversely, if A is a partitionable set of maximum size with respect to these two matroids and is partitioned so that $A = F_1 \cup F_2^*$, we may assume that F_2^* is a basis of M_2^* and $F_1 \in \mathcal{F}_1 \cap \mathcal{F}_2$.

But there is a direct algorithm for the intersection problem which works for arbitrary weights.