# NETWORK FLOWS AND COMBINATORIAL OPTIMIZATION

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# Chapter 1

# Minimum Cost Flows

The minimum cost flow problem is perhaps the most useful problem in the traditional network literature. It combines two well known special cases – the *maximum flow* and t*he shortest route problem.* The problem statement is given below:

## 1.1 Problem

Given a directed network $G = [N; A]$ and real numbers $q_i$ for each $i \in N$ and $l_{i,j}$ , $u_{i,j}$ , and $c_{i,j}$ for each $(i, j) \in A$, find arc flows $f_{i,j}$ satisfying the relations:

$$\sum_j (f_{i,j} - f_{j,i}) = q_i \; \forall \; i \in N \tag{1.1}$$

$$l_{i,j} \leq f_{i,j} \leq u_{i,j} \; \forall \; (i, j) \in A \tag{1.2}$$

$$\min \sum_{(i,j)\in A} c_{i,j} f_{i,j} \tag{1.3}$$

A necessary condition for feasibility is that $l_{i,j} \leq u_{i,j}$ for each $(i, j) \in A$ and this will be assumed to be valid through out this analysis. The usual interpretation for the data is as follows: $l$ is the *lower bound* on the arc flow; $u$ is the *upper bound*; $c$ is the *unit cost*; and $q_i$ is the *external flow* at node $i$. This is a linear program. To see that the shortest route problem is a special case let $q = (1, 0, 0..., 0, -1)$ and $c_{i,j} = d_{i,j}$ with external flows corresponding to the origin and destination respectively. (*Please note that this equivalence is "very delicate"; the meaning of this will become clear when we study shortest routes ).* To show that maximum flow problem is a special case, expand the network, if necessary, by adding the arc $(t, s)$ where $s$ is the source and $t$ is the sink. Let the lower bound be $-\infty$ and the upper bound be $\infty$ for this new arc. Let $c_{i,j} = 0$ for all the old arcs and be $-1$ for the new arc. The well known transportation problem is also a special case of this problem; in this case the network is bipartite.

There are several methods to solve min-cost flow problems. Two important ones are: (i) **the out-of-kilter algorithm** which is a primal dual type algorithm and (ii) **Klein, Jewell, Busacker & Gowan's method** that augments flow successively on the cheapest route at each step. Both of these can be rendered strongly polynomial by using a device due to **E.Tardos**. We now describe the first of these algorithms.

## 1.2 Out-of-kilter Algorithm

We will use labeling algorithms for maximum flow problem as subroutines in this algorithm. Using such an algorithm we can find a feasible flow if one exists and prove that one does not exist if that is the case. Let $f^1$ be an initial feasible flow found by such an algorithm. [Many of the flow algorithms are strongly polynomial. When $l_{i,j}$ and $u_{i,j}$ are integral (rational) then so is $f^1$ produced by these algorithms]. Since this is a primal-dual type of an algorithm, we need some of the duality results as preliminary facts for this algorithm.

### 1.2.1 Problem II:

The dual of the min-cost flow problem is:

$$x_{i,j} \geq 0; y_{i,j} \geq 0 \;\forall\; (i,j) \in A \tag{1.4}$$

$$\pi_i - \pi_j + x_{i,j} - y_{i,j} = c_{i,j} \;\forall\; (i,j) \in A \tag{1.5}$$

$$\max \sum_{i \in N} q_i \pi_i + \sum_{(i,j)\in A} (l_{i,j} x_{i,j} - u_{i,j} y_{i,j}) \tag{1.6}$$

Complementary slackness (CS) conditions for optimality are:

$$f_{i,j} > l_{i,j} \Longrightarrow x_{i,j} = 0; x_{i,j} > 0 \Longrightarrow f_{i,j} = l_{i,j} \tag{1.7}$$

$$f_{i,j} < u_{i,j} \Longrightarrow y_{i,j} = 0; y_{i,j} > 0 \Longrightarrow f_{i,j} = u_{i,j} \tag{1.8}$$

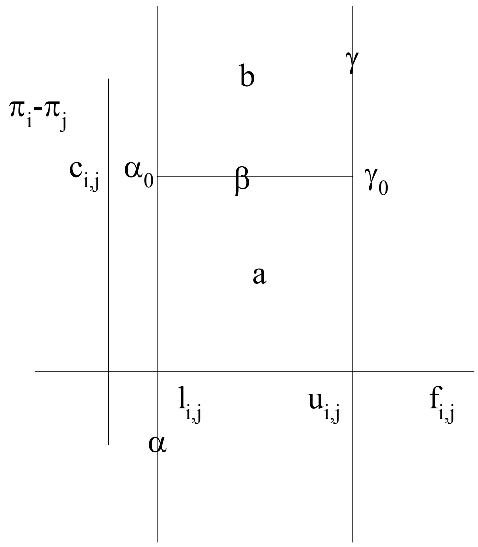
Combining these we arrive at an equivalent set on $\{\pi_i\}$ and $\{f_{i,j}\}$ :

$$\pi_i - \pi_j < c_{i,j} \Longrightarrow x_{i,j} > 0 \Longrightarrow f_{i,j} = l_{i,j} \tag{1.9}$$

$$\pi_i - \pi_j > c_{i,j} \Longrightarrow y_{i,j} > 0 \Longrightarrow f_{i,j} = u_{i,j} \tag{1.10}$$

$$l_{i,j} < f_{i,j} < u_{i,j} \Longrightarrow x_{i,j} = y_{i,j} = 0 \Longrightarrow \pi_i - \pi_j = c_{i,j} \tag{1.11}$$

These result in the well known diagram called *the complementary slackness diagram.*




Any point on the stair shaped curve satisfies CS conditions and these are the only such points. The algorithm starts with an initial feasible solution $f^1$ and any arbitrary $\pi$. [It is customary to assume that the *nature of the numbers in* $f^1$ is the same as those of $l$ and $u$; and that of $\pi$ is in the same as that in $c$. This will certainly facilitate arguments regarding finiteness, polynomialness, or strong polynomialness of the algorithms]. Given a set of values for $f$, and $\pi$ we can classify arcs corresponding to the states in the above diagram as follows;

| State | Flow | $\pi_i - \pi_j$ |
|---|---|---|
| $\beta$ | $\in (l_{i,j}, u_{i,j})$ | $c_{i,j}$ |
| $\alpha$ | $l_{i,j}$ | $< c_{i,j}$ |
| $\alpha_1$ | $l_{i,j}$ | $c_{i,j}$ |
| $\gamma$ | $u_{i,j}$ | $> c_{i,j}$ |
| $\gamma_1$ | $u_{i,j}$ | $c_{i,j}$ |
| $a$ | $\in (l_{i,j}, u_{i,j})$ | $< c_{i,j}$ |
| $b$ | $\in (l_{i,j}, u_{i,j})$ | $> c_{i,j}$ |

(1.12)

Arcs in states $\alpha$, $\alpha_1$, $\beta$, $\gamma$, $\gamma_1$, are said to be *in-kilter* and those in states $a$ and $b$ are *out-of-kilter.* If no arc is out-of-kilter, then we have an optimal solution since the CS conditions are satisfied. The algorithm *never allows an arc that is in-kilter to go out of kilter*; and it successively forces arcs that are out-of-kilter to go in-kilter. When we make changes, whether these be flow or dual variable type, the following condition must be observed: *All movement of arcs must be toward the diagram and do not cross any of the lines of the complementary slackness diagram.*

Thus, there can be no flow changes on arcs in states $\alpha$ or $\gamma$; only increases on arcs in states $\alpha_1$ and $b$; only decreases on arcs in states $\gamma_1$ and $a$ during the part of the algorithm where flow is adjusted. Change in state due to a flow change are of the following types: $b \longrightarrow \gamma$; $a \longrightarrow \alpha$; $\alpha_1 \longrightarrow \beta$; $\alpha_1 \longrightarrow \gamma_1$; $\gamma_1 \longrightarrow \beta$; $\gamma_1 \longrightarrow \alpha_1$; $\beta \longrightarrow \alpha_1$; $\beta \longrightarrow \gamma_1$. The first two result in a decrease in the number of out-of-kilter arcs.

The only possible changes in state due to dual variable changes are: $b \longrightarrow \beta$; $a \longrightarrow \beta$; $b \longrightarrow \alpha_1$; $a \longrightarrow \gamma_1$; $\gamma \longrightarrow \gamma_1$; $\gamma_1 \longrightarrow \gamma$; $\alpha \longrightarrow \alpha_1$; $\alpha_1 \longrightarrow \alpha$. The first four result in a decrease in the number of out-of-kilter arcs.

Thus, the algorithm allows flow increases on arcs in states $\beta$, $\alpha_1$, and $b$; flow decreases are permitted on arcs in states $a$, $\beta$, $\gamma_1$. Decrease in $\pi_i - \pi_j$ on arcs in states $\{b, \gamma, \alpha, \alpha_1\}$ and increase in arcs in states $\{a, \gamma, \alpha, \gamma_1\}$ are also allowed. However, increases on arcs in states $a$ and $\alpha$ as well as decreases in arcs in states $b$ and $\gamma$ are limited by the fact that we are not allowed to cross the horizontal line at $c_{i,j}$. Similar remarks apply to flow changes on these arcs as well. Now we are ready to describe the algorithm's main subroutines.

If no arc is out-of kilter then we are done. Else a major cycle of the algorithm consists in driving one of the out-of-kilter arcs into kilter by flow and dual variable changes that are performed in an alternating manner. One such cycle may transform several arcs to in-kilter states; it is guaranteed to do this for at least one such arc. If we succeed in showing that this cycle is finite (polynomial) then the whole algorithm has the same property.

Without loss we will assume that the arc under consideration is in state $a$ and is the arc $(p, q)$. In order to make this arc go in-kilter the flow changes will have to decrease flow on this arc and dual variable changes will have to increase $\pi_p - \pi_q$. We will first try to do this by flow changes only. For this purpose we consider the following maximum flow problem:

Let $p$ be the source and $q$ the sink; for arcs in state $\alpha$ let the lower and upper bound be $l_{i,j}$ ; for arcs in state $\gamma$ let both these bounds be $u_{i,j}$ ; for arcs in states $\alpha_1$, $\beta$, $\gamma_1$, let the lower bound be $l_{i,j}$ and the upper bound be $u_{i,j}$ ; for arcs in state $a$ let the lower bound be $l_{i,j}$ and the upper bound be $f_{i,j}$; for arcs in state $b$ let the lower bound be $f_{i,j}$ and upper bound be $u_{i,j}$. Starting with the current flow values that are feasible increase flow from $p$ to $q$ using circulations until either arc $(p, q)$ goes into kilter or no further flow changes are possible. If we assume a finite (polynomial) flow algorithm with real data this part of the major cycle is finite (polynomial). If we do not succeed in having arc $(p, q)$ go in kilter, then we will have a cut separating $p$ and $q$ from the max-flow problem outlined as above. Forward arcs across this cut will have flow equal to their upper bound and reverse arcs at their lower bounds. Please note that these bounds are constantly being updated for arcs in states $a$ and $b$. Since flow on arcs in state $a$ are nonincreasing and that along arcs in state $b$ are nondecreasing, and $a$ arcs must be forward and $b$ arcs must be reverse in the final cut separating $p$ and $q$ and the net flow in these arcs (the sum along $a$ arcs minus the sum along $b$ arcs) must be nonincreasing. But since all changes are made in the form of circulations the net flow remains constant. Hence the net flow on arcs in-kilter must be nondecreasing.

When we can no longer increase flows we do a dual variable change as follows:

$$\pi'_i = \begin{cases} \pi_i + \delta & i \in S \\ \pi_i & i \notin S \end{cases} \tag{1.13}$$

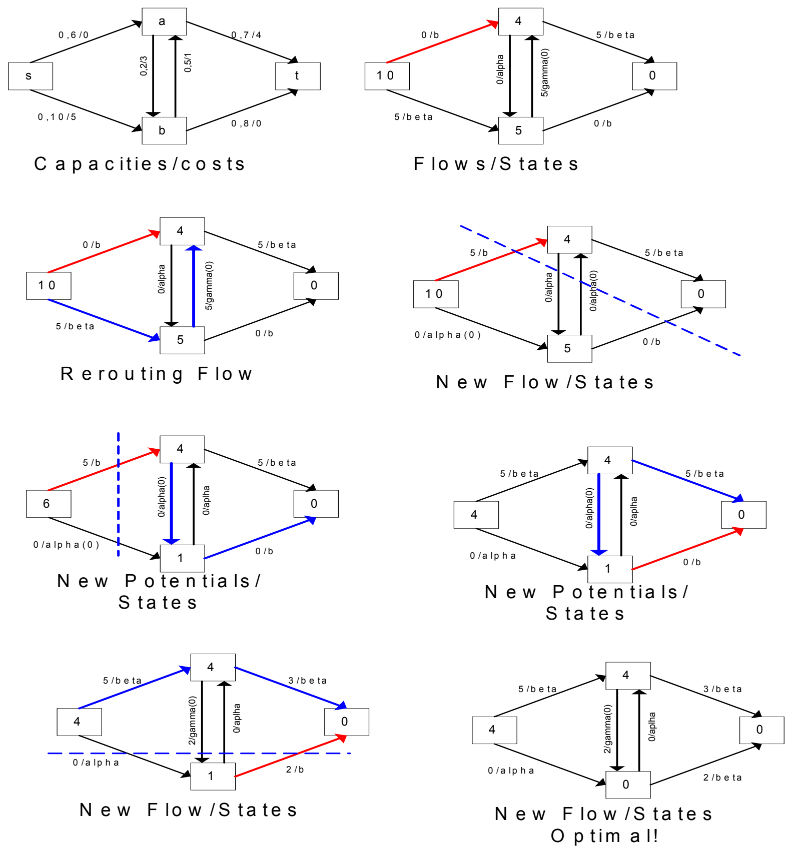
where $\delta$ is given by the relations:

$$\delta_1 = \min_{(i,j)\in A(\gamma,b)} (\pi_i - \pi_j - c_{i,j}) \tag{1.14}$$

$$\delta_2 = \min_{(i,j)\in A(\gamma,b)} (c_{i,j} - (\pi_i - \pi_j)) \tag{1.15}$$

$$\delta = \min(\delta_1, \delta_2) \tag{1.16}$$

Each time we make a dual variable change we go to the flow change part of the algorithm. When we try to do a flow change since arcs with both ends on one side of the cut (that made us do a dual variable change) do not change states they remain *admissible* for flow change if they were admissible before. The arcs across the cut either go in kilter and become admissible for flow change or change their state to one which is admissible for flow change; the only arcs that become inadmissible in this process are those in state $\alpha_1$ that are reverse and those in state $\gamma_1$ that are forward arcs. These remain in-kilter and change respectively to states $\alpha$ and $\gamma$. For example an arc in state $b$ changes to state $\beta$ or one in state $\alpha$ changes to $\alpha_1$. In any case, at least one arc across the cut becomes admissible for flow labeling and hence the set of labeled nodes enlarges. Thus, we can not do a series of dual variable changes without altering flows more than $|N|$ times. Thus, after at most a polynomial number of dual variable changes we must alter flows. An example is shown below:

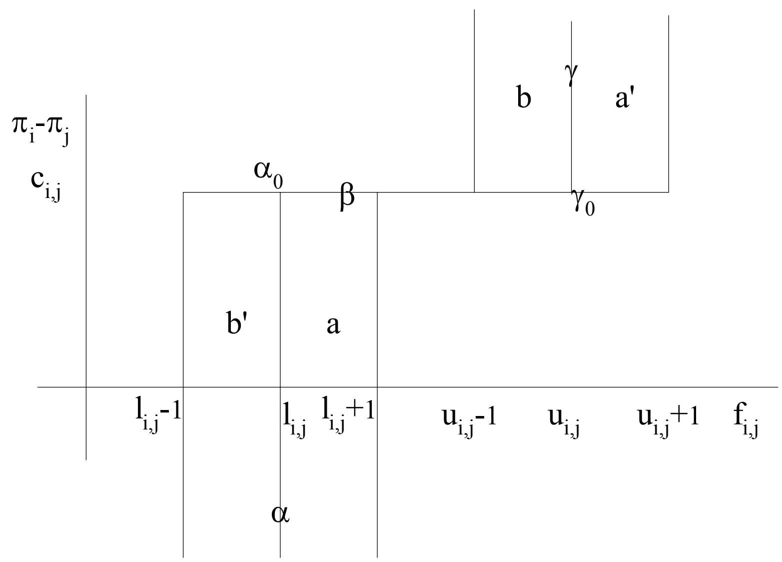



An Example of
out-of-kilter
Algorithm

Thus, if we have the same cut repeating with the same directions for the arcs without an arc going in kilter, the net flow on arcs that are in-kilter and across the cut will have changed and this means the states of these arcs can not all be the same as before. The number of possibilities for such patterns is finite and since the same pattern can not repeat the algorithm is finite. Please note that no assumption about the nature of the data (as to whether these are rational or not) is made in this analysis. *Also, if proper algorithms are used for flow changes (that are strongly polynomial) then the number of flow changes between successive dual variable changes and the number of dual variable changes between successive flow changes are both strongly polynomial. It is the number of these alternations that is not clear.* To get a polynomial algorithm for integral data (*that is not necessarily strongly polynomial)* we use scaling techniques due to **Edmonds and Karp**.

## 1.3 Edmonds—Karp Scaling

Here it is assumed that $l_{i,j}$ and $u_{i,j}$ are integral. Let $p$ be the smallest number $\ni 2^p \geq \max[|u_{i,j}|, |l_{i,j}|]$. We first need to describe a slight modification to the out-of-kilter algorithm that allows flows to violate the bounds.




*Please note that states* $\hat{a}$ and $\hat{b}$ *do not satisfy the bounds. However, the flow separation in the out-of-kilter states is no more than one unit. Hence, if a single flow change occurs then the out-of-kilter arc that started the major cycle will go in kilter. Hence the number of flow changes is strongly polynomial and hence the algorithm which starts with a flow that satisfies conservation* (*this is called a circulation*) *and a dual solution that together with the flow has every arc in one of these states will be strongly polynomial.* Also, the arcs in state $\hat{a}$ are like those in state $b$ and those in state $\hat{b}$ are like those in state $a$. Edmonds-Karp algorithm always has this property and does $p$ major cycles of this nature; each of these is strongly polynomial but the number of these major cycles depends on $p$ which in turn depends on the "size" of the data.

The only things that changes between the major cycles are the bounds. At the $k^{th}$ major cycle we solve the problem with bounds $u^k_{i,j} = \left\lceil \frac{u_{i,j}}{2^{p-k}} \right\rceil$, and $l^k_{i,j} = \left\lfloor \frac{l_{i,j}}{2^{p-k}} \right\rfloor$. We start with $l^0_{i,j}$ and $u^0_{i,j}$ and a starting flow $f^0_{i,j} = 0 \ \forall \ (i,j) \in A$. Note that these bounds at the first step are 0 or 1. Hence the starting solution together with arbitrary $\pi$ satisfy the conditions promised above. Also, since $2^{p-k}u^k \geq u$ and $2^{p-k}l^k \leq l \ \forall \ k$, if the original problem is feasible then so is starting, and every succeeding one. When $k = p$, we get the original problem. Also, the starting flow in the $(k+1)^{st}$ problem is $2(f^k)^*$. The starting $\pi$ is $(\pi^k)^*$. This set satisfies the conditions promised above as well. Hence this algorithm is polynomial (but not strongly polynomial since it depends on $p$).

Using the relations:

$$1 + \frac{u}{2^{p-k}} > u^k \geq \frac{u}{2^{p-k}}$$
$$1 + \frac{u}{2^{p-k-1}} > u^{k+1} \geq \frac{u}{2^{p-k-1}}$$

it is easy to see that:

$$-2 < u^{k+1} - 2u^k < 1$$

which implies:

$$-1 \leq u^{k+1} - 2u^k \leq 0$$

Similarly from:

$$\frac{l}{2^{p-k}} - 1 < l^k \leq \frac{l}{2^{p-k}}$$
$$\frac{l}{2^{p-k-1}} - 1 < l^{k+1} \leq \frac{l}{2^{p-k-1}}$$

it is easy to see that:

$$-1 < l^{k+1} - 2l^k < 2$$

which implies:

$$0 \leq l^{k+1} - 2l^k \leq 1$$

If

$$l^k < (f^k)^* < u^k$$

then,

$$l^{k+1} - 1 \leq 2l^k < 2(f^k)^* < 2u^k \leq u^{k+1} + 1$$

and hence $l^{k+1} \leq 2(f^k)^* \leq u^{k+1}$. Hence these arcs will continue to be in kilter (although their states may change from $\beta$ to the corner ones). We have already shown that those arcs whose flow is at the bounds may not be in kilter but the difference in flow from the kilter diagram will not be greater than one unit. Hence the algorithm is strongly polynomial at each stage and has $p$ stages; of course $p$ depends on the size of the data.

A similar algorithm that scales only the costs is due to **Rock**. Similar algorithm for "totally unimodular matrices" is found in the work of **Akgul** et al.

## 1.4 Busacker & Gowan, Jewell's Algorithm

Define the *cost of a flow augmenting path* to be the sum of the costs of forward arcs minus the sum of the costs of the reverse arcs (on which flow will be reduced and hence result in savings). The cost of a cycle is defined similarly with respect to a given orientation of the cycle.

**Lemma 1** *A feasible flow is optimal iff it admits no cycle whose cost is negative.*

**Proof**: The only if part is obvious. For the converse note that the difference between two feasible solutions is a sum of cycles. If one is less costly then one of these cycles is negative and hence the result.□

*This result is the same as the following result in LP*:

**Proposition 2**

$$x^0 \textit{ opt. to} : [\min cx : Ax = b; x \geq 0]$$

$$\Updownarrow$$

$$\exists \textit{ no } y \ni Ay = 0; y \geq -x^0; cy < 0$$

**Theorem 3** *Let $[f, F]$ be the minimum cost flow whose total value is $F$. With reference to this flow, let $P$ be a flow augmenting path with minimum cost. Let the maximum amount of additional flow that can be sent on this path be $\delta$. Then the solution that augments flow along this path by $\epsilon$ is optimal for $F + \epsilon \; \forall \; \epsilon \in [0, \delta]$.*

**Proof:** It suffices to show that there is no augmenting cycle that is negative in total cost. Let the path in the above be $P$ and the (supposed) negative cycle be denoted by $C$. Then $\exists$ an arc $(i, j) \in P \cap C$ (for otherwise the original flow is not optimal). But then $\exists$ an augmenting path $\subset P \cap C - \{(i, j)\}$ whose cost is less than that of $P$. Thus the theorem.□

The path $P$ mentioned above can be found by using a shortest path algorithm with the following *distances*:

$$d_{i,j} = \begin{cases} c_{i,j} & \text{if } f_{i,j} < u_{i,j};\; f_{j,i} = l_{j,i} \\ \min[c_{i,j}, -c_{j,i}] & \text{if } f_{i,j} < u_{i,j};\; f_{j,i} > l_{j,i} \\ -c_{j,i} & \text{if } f_{i,j} = u_{i,j};\; f_{j,i} > l_{j,i} \\ \infty & \text{if } f_{i,j} = u_{i,j};\; f_{j,i} = l_{j,i} \end{cases}$$

The shortest path corresponds to the minimum cost augmenting path and negative cycle corresponds to an improving cycle on the previous flow. If all the original costs are nonnegative then starting with zero flows the first path is on positive distances. **Edmonds and Karp** showed a method by which we can always keep it this way by using $d^*_{i,j}$ instead of $d_{i,j}$ mentioned above. $d^0_{i,j} = c_{i,j}$ and $f_{i,j} = 0$. Thereafter, let

$u^k_i$ =the length of the shortest path from $s$ to $i$ with respect arc lengths $d^*_{i,j}$ and

$$d^*_{i,j} = d_{i,j} + \pi^k_i - \pi^k_j$$
$$\pi^{k+1}_i = \pi^k_i + u^k_i$$
$$\pi^0_i = 0$$

where $d_{i,j}$ is as defined above with respect to a given flow. These distances are nonnegative since they are like *reduced costs* at optimality and the shortest path with the two distances are the same since the total distance is modified by a constant for all paths. This sort of an algorithm is well suited to parametric analysis.

## 1.5 Strongly Polynomial Algorithms

### 1.5.1 Eva Tardos' Modification to Out-of-kilter Algorithm:

A slight modification of the minimum cost flow problem is considered (without any loss of generality):

$$\min \textstyle\sum_{ij} c_{i,j} f_{i,j}$$
$$\textstyle\sum_j (f_{i,j} - f_{j,i}) = 0 \ \forall \ i \in N$$
$$l_{i,j} \leq f_{i,j} \leq u_{i,j}$$

No assumption is made regarding the nature of the data. The set of feasible solutions is denoted by $P(l, u)$. A solution which satisfies the conservation condition is called a *circulation* and the set of circulations is a vector space. An arc $(i, j)$ is said to be *tight* if $l_{i,j} = u_{i,j}$; let the set of tight arcs be denoted by $T(l, u)$. Two cost vectors $c$ and $c'$ are said to be $(l, u)$ *equivalent* if for every nontight arc $(i, j)$ the relation: $c'_{i,j} = c_{i,j} - \pi_i + \pi_j$ holds for some *potential* function $\pi$ on the nodes. This term is appropriate because:

**Lemma 4** *If $c$ and $c'$ are $(l, u)$ equivalent then the set of optimal solutions to the problem is the same for both cost functions.*

**Proof:**

$$c'f = cf - \sum_i \sum_j f_{i,j}(\pi_i - \pi_j) + \sum_{(i,j) \in T(l,u)} f_{i,j}(c'_{i,j} - c_{i,j} + \pi_i - \pi_j)$$

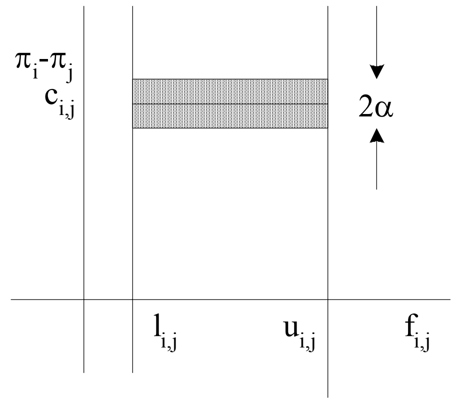
The last term is a constant independent of $f$ and since $f$ is a circulation the middle term is zero (*Note that $f$ can be written as a sum of simple cycle flows by the peeling process described before and hence the second term can be written as the sum of terms each of which is of the form $\epsilon \sum_i \sum_j (\pi_i - \pi_j)$ where the sum is on the arcs of a cycle and hence is zero*). Equivalently,

$$\sum_i \sum_j f_{i,j}(\pi_i - \pi_j) = \sum_i \pi_i \sum_j (f_{i,j} - f_{j,i}) = 0$$

since $f$ is a circulation. Hence $c'f - cf$ is a constant independent of $f$ and hence the result.□

While we may take the $c'_{i,j}$ to be arbitrary for arcs that are tight, the particular choice of $c'$ that is a circulation and has $c'_{i,j} = 0$ for tight arcs is of special interest to us and is said to *fit* $P(l, u)$. *That there are such vectors will be shown later.*

Now we need to introduce the notion of "*almost complementary slackness*".




Let $\alpha > 0$. Let $(f, \pi)$ be a pair $\ni$

$$c_{i,j} - \pi_i + \pi_j > \alpha \Longrightarrow f_{i,j} = l_{i,j}; \pi_i - \pi_j - c_{i,j} > \alpha \Longrightarrow f_{i,j} = u_{i,j}$$

We say that the pair $(f, \pi)$ satisfies *almost complementary slackness condition.* Of course such a solution need not be optimal for the problem with $c$ as the cost function. However, we may "justify" our statement of "almost" by the following result:

**Theorem 5** *If $(f, \pi)$ are such a pair as above and $f^*$ is any optimal solution then:*

$$|c_{i,j} - \pi_i + \pi_j| \geq |N|\,\alpha \Longrightarrow f^*_{i,j} = f_{i,j}$$

*( Note that the flow is at a bound on these arcs).*

**Proof:** Suppose this condition is violated on some arc say $(i, j)$ for $f^*$.
**Claim**: We may assume without loss that $f^* \geq f$ :
**Proof**: If $f^*_{k,l} < f_{k,l}$ for some arc $(k, l)$ then we reverse the direction of the arc and let the new values for $f_{k,l}, f^*_{k,l}, l_{k,l}, u_{k,l}, c_{k,l}$, be given respectively *by* $- f_{k,l}$, $-f^*_{k,l}$, $-u_{k,l}$, $-l_{k,l}$, and $-c_{k,l}$. After we do this for every such arc the claim is true and the problem is unaffected by this transformation.□

Since for arc $(i, j)$, $|c_{i,j} - \pi_i + \pi_j| \geq |N|\,\alpha$, we can have two cases. (i) the term in the absolute value sign is negative and hence $c_{i,j} - \pi_i + \pi_j < -\alpha$ and this implies that $u_{i,j} = f_{i,j} \leq f^*_{i,j} \leq u_{i,j}$ and hence $f_{i,j} = f^*_{i,j}$ and this arc would not violate the condition of the theorem. Hence we must have the other case that $c_{i,j} - \pi_i + \pi_j \geq |N|\,\alpha$ and hence $f_{i,j} = l_{i,j}$. Since both $f$ and $f^*$ are circulations so is $f^* - f \geq 0$. Thus $\exists$ a cycle $C \ni (i, j) \in C$ and $f^* - f$ is positive on this cycle. Let the minimum value of $f^* - f$ on this cycle be $\epsilon$. Let $\hat{f}$ be the flow obtained by decreasing from $f^*$ the flow on arcs on this cycle by $\epsilon$. Then

$f \leq \hat{f} \leq f^*$ and hence $\hat{f}$ is feasible with $(l, u)$ as bounds. We will show that under the conditions mentioned above $\hat{f}$ is better than $f^*$.

$$(k,l) \in C \Longrightarrow f_{k,l} < f^*_{k,l} \leq u_{k,l}$$

$$\Downarrow$$

$$\pi_k - \pi_l - c_{k,l} \leq \alpha$$

$$\Updownarrow$$

$$c_{k,l} - \pi_k + \pi_l \geq -\alpha$$

$$\begin{aligned} c\hat{f} &= c\hat{f} = cf^* - \epsilon \sum_{(k,l)\in C} c_{k,l} \\ &= cf^* - \epsilon \sum_{(k,l)\in C} (c_{k,l} - \pi_k + \pi_l) \qquad (1.17) \\ \leq &= cf^* - \epsilon[|N|\, \alpha - (|C| - 1)\alpha] < cf^* \qquad (1.18) \end{aligned}$$

$c\hat{f} = cf^* - \epsilon \sum_{(k,l)\in C} c_{k,l} = cf^* - \epsilon \sum_{(k,l)\in C}(c_{k,l} - \pi_k + \pi_l)$
$\leq cf^* - \epsilon[|N|\, \alpha - (|C| - 1)\alpha] < cf^*$
This contradicts the supposition that $f^*$ is optimal. Hence the result.□

**Tardos' Algorithm**

**Input**: $G = [N; A]$; $l_{i,j}$, $u_{i,j}$, $c_{i,j} \forall\ (i,j) \in A$ with $l \leq u$.
**Output**: $\hat{l}_{i,j}$, $\hat{u}_{i,j} \forall\ (i,j) \in A$ satisfying $l \leq \hat{l} \leq \hat{u} \leq u$ and $P(\hat{l}, \hat{u})$ is the set of optimal solutions to the original problem.
**Step 1**: Check if $c$ is $(l, u)$-equivalent to 0. If the answer is yes, stop; $\hat{l} = l$; and $\hat{u} = u$. *Note that this check is strongly polynomial.* Else go to step 2.
**Step 2**: Let $T = \{(i,j) : l_{i,j} = u_{i,j}\}$. Let $E^{\bar{T}}$ be the node arc incidence matrix of the graph $G^{\bar{T}} = [N; \bar{T}]$ with redundant rows removed. The rows of $E^{\bar{T}}$ are linearly independent. (This is accomplished by removing a row corresponding to one node from each component of the corresponding graph $G^{\bar{T}}$) Let $c^*$ be defined by the relations:
$c^* = [I - \hat{E}^t(\hat{E}\hat{E}^t)^{-1}\hat{E}]c$ where $\hat{E} = E^{\bar{T}}$.

Using Lagrange's method of unknown multipliers we can show that *this vector minimizes* $\|c - d\|$ *over the set* $\{d : Ed = 0; d_T = 0\}$. *Thus* $c^*$ *"fits"* $P(l, u)$. Hence $c^*$ is $(l, u)$-equivalent to $c$. Note also that $c^*$ can be computed by a strongly polynomial algorithm. *We are "merely" solving a system of linear equations and the variables may take on negative values.* Let $c' = \lambda c^*$ for some $\lambda > 0 \ni \max_{(i,j)\in A} \left|c'_{ij}\right| = \left\lceil |N| \cdot \sqrt{|A|} \right\rceil e$. Let $\bar{c} = \lceil c' \rceil$. Use the out-of kilter algorithm (the version of which uses strongly polynomial algorithm for flow changes) on the problem with $\bar{c}$ as the cost function to get an optimal pair $(f, \pi)$. This satisfies the complementary slackness for $\bar{c}$. It may violate these for $c'$, but by no more than a fraction on any arc. *Since $c^*$ is $(l,u)$-equivalent to $c$ and $c'$ is positive multiple of $c^*$ the set of optimal solutions for $c$ and $c'$ are the same.* Thus, $(f, \pi)$ satisfy "almost" complementarity conditions for the problem whose objective function is $c'$ with $\alpha < 1$. Hence if we find an arc $(i, j)$ that is not currently "tight" satisfying the relations: $|c_{i,j} - \pi_i + \pi_j| \geq |N|$ (and hence $\geq |N| \alpha$) then we can make this arc tight (for $c'$ and hence $c$) by the previous results. That such an arc exists needs to shown as well as the fact the set of arcs that were tight will continue to be tight.

**Step 3**: Setting new bounds:

If $\left|c'_{i,j} - \pi_i + \pi_j\right| < |N|$, let $l^{new}_{i,j} = l^{old}_{i,j}$ and $u^{new}_{i,j} = u^{old}_{i,j}$. Else let $l^{new}_{i,j} = u^{new}_{i,j} = f_{i,j}$. *Such arcs are the newly created "tight" arcs.* We will show that the "old" tight arcs are still tight and at least one more arc will be tight.

That the "old" tight arcs are still tight follows from the fact that for these arcs either definition of the new bounds makes them tight. We have already shown that all optimal solutions to the original problem are feasible to the new problem and hence optimal to it i.e. $P_{opt} \subset P(l^{new}, u^{new})$ where $P_{opt}$ is the set of optimal solutions to the original problem.

**Theorem 6** $T(l^{new}, u^{new}) \subset T(l^{old}, u^{old})$.

**Proof**:All we need to show is that at least one more arc is tight at each step. For this we need to show that at least for one arc $(i, j)$ that is *not tight with the old bounds* satisfies the condition: $\left|c'_{i,j} - \pi_i + \pi_j\right| > |N|$. For this purpose let:

$$c^{"}_{i,j} = \begin{cases} c'_{i,j} - \pi_i + \pi_j & (i,j) \notin T(l^{old}, u^{old}) \\ 0 & else \end{cases}$$

We want to show that $\max_{(i,j)\in A} \left|c^{"}_{i,j}\right| \geq |N|$. $c'$ "fits" $P(l, u)$; recall $c'$ is a circulation and $c'_{i,j} = 0$ if $(i, j) \in T(l^{old}, u^{old})$.

**Claim**: $c' \perp (c^{"} - c')$

**Proof**: $\sum_{(i,j)\in A} c'_{i,j}(c^{"}_{i,j} - c'_{i,j}) = \sum_i \sum_j c'_{i,j}(\pi_j - \pi_i) = -\sum_i \pi_i \sum_j (c'_{i,j} - c'_{j,i}) = 0$

since $c'$ is a circulation. Hence $c'$, $c^{"}$, and $c^{"} - c'$ form a right angle triangle with $c^{"}$ as the hypotenuse and hence $\left\|c^{"}\right\| \geq \|c'\|$. Hence

$\max\left|c^{"}_{i,j}\right| \geq (1/\sqrt{|A|}) \left\|c^{"}\right\| \geq (1/\sqrt{|A|}) \|c'\| \geq (1/\sqrt{|A|}) \max\left|c'_{i,j}\right|$

$= \left\lceil (|N| \sqrt{|A|})/\sqrt{|A|}) \right\rceil = |N|$.

This completes the description of **Eva Tardos**' algorithm for minimum cost flows which is strongly polynomial. There are now several strongly polynomial algorithms with varying properties and a good source for these is **"Network Flows"** by **R.K.Ahuja, T.L.Magananti, and J.B.Orlin, August 1988**. The main promising candidates for strongly polynomial algorithms are:

**J.B.Orlin**: Genuinely Polynomial Simplex and Nonsimplex Algorithms for Minimum Cost Flow Problem, MIT Report# 1615-84, 1984.

**S.Fujishige:** An $O(m \log\ n)$ Capacity Rounding Algorithm for the Minimum Cost Circulation Problem: A Dual Framework of Tardos' Algorithm, *Math. Prog.*35 (1986), 298-309.

**A.V. Goldberg and R.E. Tarjan:**Finding Minimum Cost Circulation by Canceling Negative Cycles, Proc. $20^{th}$ Symp. on Theory of Computing, 1988.

*In general see these proceedings from 86-88 under these names.*

### 1.5.2 Golberg-Tarjan Algorithm:

Just as Tardos' algorithm is a modification of out-of-kilter algorithm so is Goldberg-Tarjan a modification of Busacker-Gowen-Jewell-Klein algorithm. It finds the improvement on negative cycles that minimize the ratio of cost change to the number of arcs in the cycle. That such a cycle can be found by a strongly polynomial algorithm is well known in the literature of ratio function problems.

**Problem:** Given a directed graph $G = [N; A]$ with the additional property that $(i, j) \in A \Longrightarrow (j, i) \in A$ — the graph is *symmetric* and $c_{i,j} = -c_{j,i}$ and each arc has a capacity $u_{i,j}$ (*apparently* $u_{j,i}$ *need not have any relationship to* $u_{i,j}$). We want a flow $f_{i,j}$ satisfying the relations:

$$\begin{array}{c} f_{i,j} = -f_{j,i} \\ \sum_j f_{i,j} = 0 \;\forall\; i \in N \\ \min(cf/2). \end{array}$$

The *residual capacity* of an arc $(i, j)$ with respect to flow $f$ is defined by $u^f_{i,j} = u_{i,j} - f_{i,j}$ and an arc is said to be a *residual arc* if this is positive. A nonresidual arc is said to be *saturated.* $A^f$ is the set of residual arcs with respect to $f$. A *residual cycle* is a simple cycle of residual arcs. A residual cycle is *negative* if its cost is negative and the *capacity* of a residual cycle is the minimum of the residual capacities of the arcs on that cycle. $c^\pi_{i,j} = c_{i,j} - \pi_i + \pi_j$ where $\pi$ is a potential function. A circulation $f$ is optimal iff $\exists$ a potential $\pi$ satisfying the relation: $u^f_{i,j} > 0 \Longrightarrow c^\pi_{i,j} \geq 0$. These are the complementary slackness conditions. For an $\epsilon \geq 0$, a circulation $f$ is said to be *$\epsilon$-optimal* if $\exists$ a potential function $\pi \ni u^f_{i,j} > 0 \Longrightarrow c^\pi_{i,j} \geq -\epsilon$ (*sometimes to be more precise we may say that the pair* $(f, \pi)$ *is $\epsilon$-optimal rather than simply* $f$ *is $\epsilon$-optimal*). $\epsilon$-optimality of the pair $(f, \pi)$ implies $\epsilon'$-optimality for $\epsilon' \geq \epsilon$, assuming that $\epsilon \geq 0$. If a circulation is $\epsilon$-optimal for "small enough" $\epsilon$ then it is optimal by:

**Theorem 7** *If all* $c_{i,j}$ *are integral, and* $\epsilon < 1/|N|$*, then any $\epsilon$-optimal solution is optimal.*

**Proof**: Let $(f, \pi)$ be $\epsilon$-optimal. Let $G^f = [N; A^f]$ and let $C$ be a simple cycle in $G^f$. The reduced cost of the cycle is at least $-|C|\,\epsilon$ which in turn is at least

$-|N|\epsilon > -1$. The reduced cost of the cycle is equal to the original cost which must be integral and hence nonnegative. Hence $f$ is optimal by $B-G-J$.□

*This is an important qualitative difference between Tardos' exposition. Whether Goldberg actually uses this theorem and needs it is not clear.*

Let:

$$-\epsilon(f,\pi) = \min_{(i,j):u^f_{i,j}>0} c^\pi_{i,j}$$

$$\epsilon(f) = \min_\pi \epsilon(f,\pi)$$

Then $\epsilon(f)$ is the smallest $\epsilon$ such that $f$ is $\epsilon$-optimal. Let $\mu(f)$ equal the mean cost of the minimum mean cost residual cycle. Then:

**Theorem 8** *For any circulation $f$, $\epsilon(f) = \max[0, -\mu(f)]$.*

**Proof**: Let $C$ be a cycle in $G^f$ and let $|C| = p$. For any $\epsilon \geq 0$, let $c^\epsilon$ be the vector whose components are $c^\epsilon_{i,j} = c_{i,j} + \epsilon$. If $f$ is $\epsilon(f)$-optimal,

$$0 \leq \sum_{(i,j)\in C} c^{\epsilon(f)}_{i,j} = p\epsilon(f) + \sum_{(i,j)\in C} c_{i,j}$$

hence mean cost of $C$ is $\geq -\epsilon(f)$ and hence the result $\mu(f) \geq -\epsilon(f)$.

Conversely, let $C$ be the cycle that has the minimum value for the mean cost of the residual cycles and let this value be $\mu(f)$. If $\mu(f) \geq 0$, then $\exists$ no negative cycle and hence $\epsilon(f) = 0 \geq -\mu(f)$. Since $\mu(f)$ is the minimum mean cost cycle in $G^f$, the minimum cycle in $G^f$ with costs of arc $(i,j)$ modified to $\hat{c}_{i,j} = c_{i,j} - \mu(f)$ is $\geq 0$. Hence $f$ is optimal for $\hat{c}$. Hence it is $-\mu(f)$-optimal for $c$. Hence $\epsilon(f) \leq -\mu(f)$. This completes the lemma.□

For a given pair $(f,\pi)$, *the set of admissible arcs* $A^{(f,\pi)}$ is the set of arcs with $c^\pi_{i,j} < 0$. $G^{(f,\pi)} = [N, A^{(f,\pi)}]$ is called *the admissible graph.*

**Lemma 9** *Let $(f,\pi)$ be $\epsilon$-optimal and let $G^{(f,\pi)}$ be acyclic. Then $f$ is $(1-\frac{1}{|N|})\epsilon$-optimal.*

**Proof: Let** $C$ be a simple cycle in $G^f$ and let the number of arcs in $C$ be $p$. Since $f$ is $\epsilon$-optimal, the cost of each arc on $C$ is $\geq -\epsilon$. Since $G^{(f,\pi)}$ is acyclic, at least one arc in $C$ has nonnegative cost.. Hence mean cost of $C$ is at least $-(p-1)\epsilon/p \leq -(|N|-1)\epsilon/|N| = -(1-\frac{1}{|N|})\epsilon$. Hence the lemma.□

**Lemma 10** *Canceling a minimum mean cycle can not increase $\epsilon(f)$.*

**Proof**: Let $C$ be such a cycle. Before $C$ is canceled, by $\epsilon(f)$-optimality of $(f,\pi)$ we have $c^\pi_{i,j} \geq -\epsilon(f)$ for all residual arcs. Since we would attempt to cancel only if $\epsilon(f) \neq 0$, it follows that $\mu(f) = -\epsilon(f)$ and hence every arc on $C$ satisfies $c^\pi_{i,j} = -\epsilon(f)$. Since canceling $C$ creates only new arcs of the type that is a reversal of an arc in $C$, such arcs have cost equal to $\epsilon(f)$ by antisymmetry. Hence after canceling $C$ all residual arcs satisfy $c^\pi_{i,j} \geq -\epsilon(f)$ where $(f,\pi)$ is the pair before canceling. Hence after canceling the value of $\epsilon(f^{new}) \leq \epsilon(f^{old})$.

**Lemma 11** *A sequence of at most $|A|$ minimum mean cycle cancellations reduces the value of $\epsilon(f)$ by a factor of $(1 - \frac{1}{|N|})$.*

**Proof:**Let the initial value (before the sequence) be $\epsilon(f)$ and let the pair of solutions be $(f, \pi)$. Clearly $c^{\pi}_{i,j} \geq -\epsilon(f)$ for all residual and hence admissible arcs. Canceling a cycle that consists only of admissible arcs adds only residual arcs with positive reduced cost and deletes at least one admissible arc. Hence there can be no more than $|A|$ such consecutive iterations before we have no admissible arcs and hence an optimal $f$ with $\epsilon = 0$. If at any stage an arc of the canceled cycle $C$ has an arc which is not admissible and hence has positive reduced cost, by lemma the mean cost of $C$ is $\geq -(1 - \frac{1}{|N|})\epsilon(f')$. Hence just before canceling this cycle, $\epsilon \leq (1 - \frac{1}{|N|})\epsilon$ and since $\epsilon(f)$ never increases the result follows.□

**Theorem 12** *If all costs are integral and* $\max\ |c_{i,j}| = K$*, minimum mean cycle canceling algorithm takes no more than $O(|N|\,|A|\log(|N|\,K))$ iterations.*

**Proof**: Follows from the fact that the initial solution has $\epsilon \leq K$ and the final one has $\epsilon < 1/|N|$ and in no more than $|A|$ iterations the $\epsilon$ reduces by a factor of $(1 - \frac{1}{|N|})$.

*Please note that the algorithm is now polynomial but not strongly so. This does to the* **B-G-J** *algorithm what* **Edmonds-Karp** *did to the out-of-kilter algorithm.*

An arc is said to be *$\epsilon$-fixed* if for all $\epsilon$-optimal solutions the flow on this arc is the same. *Recall that Tardos' approach was similar.* The following is a slight generalization of Tardos' result.

**Theorem 13** *Let $(f, \pi)$ be $\epsilon$-optimal for some $\epsilon > 0$ and let $\left|c^{\pi}_{i,j}\right| \geq 2\,|N|\,\epsilon$. Then $(i, j)$ is $\epsilon$-fixed.*

**Proof**: By antisymmetry, it suffices to prove the case where $c^{\pi}_{i,j} \geq 2\,|N|\,\epsilon$. Let $f'$ be a circulation with $f'_{i,j} \neq f_{i,j}$. Since $c^{\pi}_{i,j} > \epsilon$, $f_{i,j}$ must be the least value possible and hence $f_{i,j} = -u_{j,i}$ and therefore $f'_{i,j} > f_{i,j}$. Consider $G_{>} = [N; A_{>}]$ where $A_{>}$ is the set of arcs with flow $f' > f$. Since $f$ and $f'$ are circulations, $\exists$ a cycle $C$ passing through $(i, j)$ in $G_{>}$ [this arc is certainly in $G_{>}$]. Let $p$ be the number of arcs in $C$ and since all arcs of $C$ are residual arcs in $f$, the cost of $C$ is at least $c^{\pi}_{i,j} - (p-1)\epsilon > |N|\,\epsilon$. Now consider the cycle $\bar{C}$ obtained by reversing all arcs on $C$. Note that this is a cycle in $G_{<}$ (defined similar to $G_{>}$) and hence is a cycle of $G^{f'}$. By antisymmetry, the cost of $\bar{C} < -\,|N|\,\epsilon$ and hence its mean cost is $< -\epsilon$. Hence $f'$ is not $\epsilon$-optimal. Hence the theorem.□

*Note that the theorem not only asserts the flow is fixed but also that it is fixed at the value in $f$ and at not only the next step but thereafter because $\epsilon$ is nonincreasing.*

**Theorem 14** *The minimum mean cycle canceling algorithm terminates in no more than* $O(|N|\,|A|^2 \log |N|)$ *iterations.*

**Proof**: We will use the relation: $(1 - \frac{1}{n})^{n(\ln\ n+1)} \leq \frac{1}{2n}$ for $n \geq 2$. Let $k = |A|\,(|N|\ \lceil \ln |N| + 1 \rceil)$. Divide the iterations into groups of $k$ consecutive iterations. Claim that after each group, a new arc gets fixed. The theorem follows from this. We now prove the claim.

Let the flows before and after this many iterations be $f$ and $f'$ and let $\epsilon(f)$ and $\epsilon(f')$ be $\epsilon$ and $\epsilon'$ respectively. It is clear from the previous results that $\epsilon' \leq \frac{\epsilon}{2|N|}$. Let the first cycle cancelled in this group be $C$. Since the mean cost of $C$ is $-\epsilon$, $\exists$ an arc say $(i, j)$ on $C \ni c^{\pi}_{i,j} \leq -\epsilon \leq -2\,|N|\,\epsilon'$. By lemma, the flow on this arc will not change after this group of iterations. Note that the flow on this arc *is* changed in this group and hence this is additional arc on which the flow will no longer change as claimed.□

**Theorem 15** *Finding a minimum mean cost cycle is solvable by a strongly polynomial algorithm.*

**Proof**: See any of C, K, M.

### 1.5.3 Fujishighe's Version of Tardos' Algorithm

**Fujishighe** gives another approach which avoids the "unnatural" operation of projection of the cost vector into the circulation space (*or more precisely the orthogonal complement of the "tension" space – space of potential functions*).

### 1.5.4 Algorithm F:

**Definition 16** $T(l.u) = [(i, j) \in A : l_{i,j} = u_{i,j}]$ : *the set of* tight *arcs.*

**Input**: $G = [N, A]; l_{i,j} \leq u_{i,j}$; and $c_{i,j}$ for each arc $(i, j) \in A$.
**Output**: $\hat{l}_{i,j}, \hat{u}_{i,j}$ for each arc $(i, j)$ satisfying $l \leq \hat{l} \leq \hat{u} \leq u$ where $P(\hat{l},\ \hat{u})$ is the set of all optimal solutions to the min-cost flow problem with $l$ and $u$ as the bounds and $c$ as the cost vector.
**Step 0**: $t = 0; l^0 = l; u^0 = u$.
**Step 1**: Find a *base (principal forest)* of nontight arcs and extend it to a base of the original graph $G$. Call this base (of the graph $G$) $\tau$. Let $v^0$ be a root (arbitrarily selected) and let $\pi_i$ be the *length* of the unique path from $v^0$ to $i$ in $\tau$.
**Step 2**: Let $M^t = \max_{(i,j)\notin T(l',u')} \left| c^{\pi}_{i,j} \right|$

If $M^t = 0$, we are done; *in Tardos' terminology*, $c$ *and* $0$ *are* $(l^t$,$u^t)$ — *equivalent*. Stop; output $\hat{l} = l^t; \hat{u} = u^t$. Else let $(p^t, q^t)$ be the arc that attains $M^t$; *please note that this arc is nontight.*

Let $\tilde{c}_{i,j} = \left\lceil (c^{\pi}_{i,j}\,|N|^2)/M^t \right\rceil\ \ \forall\ (i, j) \in A$. Using $\tilde{c}_{i,j}$ as cost with $l = l^t$ and $u = u^t$ solve the mincost flow problem (with a strongly polynomial flow algorithm). This will be strongly polynomial and result in a pair of optimal solutions $[\tilde{f}, \tilde{\pi}]$ to the primal and dual.

If $|\tilde{c}_{i,j} - \tilde{\pi}_i + \tilde{\pi}_j| < |N|$, let $l^t_{i,j} = l^t_{i,j}$ and $u^t_{i,j} = u^t_{i,j}$. Else let $l^t_{i,j} = u^t_{i,j} = \tilde{f}_{i,j}$. *Such arcs are the newly created "tight" arcs.* Increment $t$ by 1 and go to step 1.

It is easy to see that the "old" tight arcs are still tight. To show that at least one more arc will be tight, we observe that:

Let $C$ be any circuit formed by $\tau \cup \{(p^t, q^t)\}$. Length of $C$ relative to $\tilde{c}_{i,j} = \pm |N|^2$ and *this is invariant under any other* $\pi$. Hence $\forall\ \tilde{\pi},\ \sum_{(i,j)\in C} c^{\tilde{\pi}}_{i,j} = \pm |N|^2$. Hence $\exists$ an arc of this circuit with $\left|c^{\tilde{\pi}}_{i,j}\right| \geq |N|$ as required. Please note that such an arc was not tight before.

This avoids the use of projection used in Tardos' paper.

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