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Multicommodity Maximum Flow Problems

Apparently the first work dealing with multicommodity flow problems is due to Jewell [J] and [FF] in 1958. The formulation is rather straight forward in the directed case; the undirected case has several possibilities. In some of these problems, there may be capacities only on the total flow on an arc. Some cases have individual capacities for each commodity on each arc and some have both types of capacities. An important point has to be made at the outset: *it is not the just fact that there are apples and oranges flowing that gives rise to many commodities in the mathematical format*; *nor is it solely due to different origin — destination pairs as mathematically oriented writers might have suggested. It is due to the combination of both these factors. More precisely, it is due to individual and common capacity constraints for the different commodities on each arc.* The most general format for directed graphs is a linear program of the form:

$$\sum_j f^k_{i,j} - \sum_j f^k_{j,i} = \begin{cases} F_k & i = s_k \\ 0 & i \neq s_k, t_k \\ -F_k & i = t_k \end{cases} \tag{1.1}$$

$$l^k_{i,j} \leq f^k_{i,j} \leq u^k_{i,j}; l_{i,j} \leq \sum_k f^k_{i,j} \leq u_{i,j} \tag{1.2}$$

$$\max \sum_k F_k \tag{1.3}$$

In this formulation, k refers to the commodity and we refer to $f^k_{i,j}$ as k-flow on arc (i, j). Certainly, this is a linear program and can therefore be solved by LP methods. Also, we know from recent advances in LP (for example, [K], [NK], and [T]) that this problem has a strongly polynomial algorithm. A *strongly polynomial algorithm* in this case means that the effort is bounded by a polynomial function of the number of nodes. However, we do lose the integrality property even when the number of commodities is two. Also, we can no longer assert that maximum flow equals minimum cut. *In case* $s_k = s \,\forall\, k$, *we can introduce a supersink* T *which is connected to all the destination and we have the same destination as well. In this case we can combine all commodities if there are no individual capacities for each commodity and thus reduce the problem to one of single commodity.* Also, we may assume without loss that $\sum_k l_{i,j}^k \le l_{i,j} \le u_{i,j} \le \sum_k u_{i,j}^k$. There is not much more work done on the directed case except to solve this LP by methods of large scale linear programming such as decomposition theory or generalized upper bounding (see [D]) or column generation techniques such as those of [J] or [FF].

In the undirected case, the formulation most often used is the following:

$$\sum_j f_{i,j}^k = \begin{cases} F_k & i = s_k \\ 0 & i \ne s_k, t_k \\ -F_k & i = t_k \end{cases} \tag{1.4}$$

$$0 \le \left| f_{i,j}^k \right| \le u_{i,j}^k; 0 \le \sum_k \left| f_{i,j}^k \right| \le u_{i,j} \tag{1.5}$$

$$\max \sum_k |F_k| \tag{1.6}$$

where all flows are algebraic (meaning positive value for $f_{i,j}^k$ implies that the commodity k flows in the direction from i to j and negative flow in the reverse direction). Even in this case, in most papers, there is no bound for individual commodities. With this last assumption, we can assume that there are at least two distinct origins and at least two distinct destinations. The role of origins and destinations in undirected networks is symmetric. We are now ready to begin the discussion of the first important contribution in this area due to **T.C. Hu** for $k = 2$. *The network is undirected and there are no edge capacities for individual commodities.*

Definition 1 *A* disconnecting set (cut) *is a set of edges whose removal leaves no path between* s_k *and* t_k *for* $\forall\, k$.

One could rewrite the capacity constraint as follows:

$$-u_{i,j} \le f_{i,j}^1 + f_{i,j}^2 \le u_{i,j} \forall\, (i,j) \in E \tag{1.7}$$

$$-u_{i,j} \le f_{i,j}^1 - f_{i,j}^2 \le u_{i,j} \forall\, (i,j) \in E \tag{1.8}$$

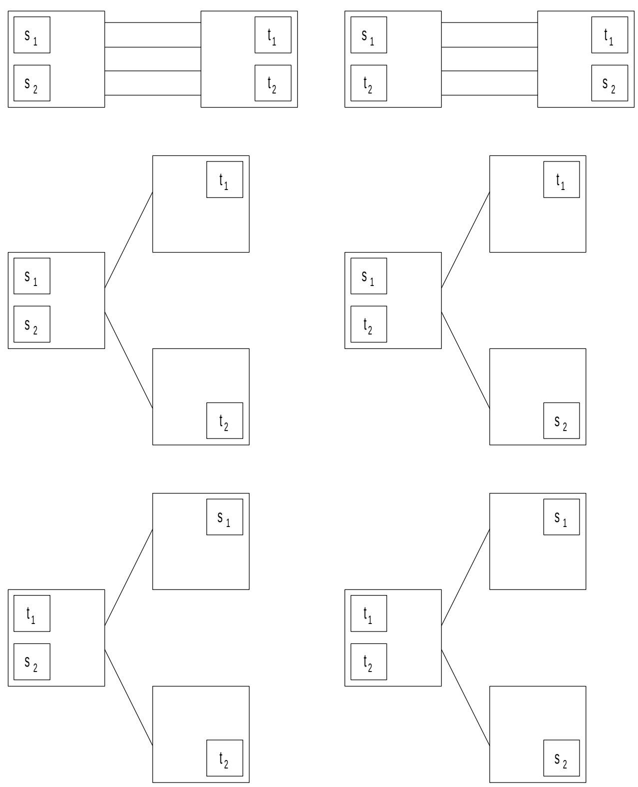
Lemma 1 *The removal of a disconnecting set of arcs for k pairs of nodes will result in a network with at most $k+1$ more components than that of the original graph.*

Proof: Remove the arcs of this set one by one in any order. Any time the number of components increases (it increases by one) at least one more (new) pair gets disconnected. Hence the result.□

Let $c(s_1, s_{2,\ldots,}s_k; t_1, t_2, \ldots, t_k)$ denote the minimum value of the disconnecting set that separates these source sink pairs. Let $[\{\mathbf{S}\}]$ denote that the set of nodes in {S} have been condensed into a single node.

Lemma 2 $c(s_1, s_2; t_1, t_2) = \min[c([\{\mathbf{s_1}, \mathbf{s_2}\}], [\{\mathbf{t_1}, \mathbf{t_2}\}]), c([\{\mathbf{s_1}, \mathbf{t_2}\}], [\{\mathbf{s_2}, \mathbf{t_1}\}])]$.

Proof: Since the above disconnecting set produces at most three components, only the following situations are possible:



The formulas apply in each case.

Lemma 3 F_1, F_2 *are feasible flows only if:*

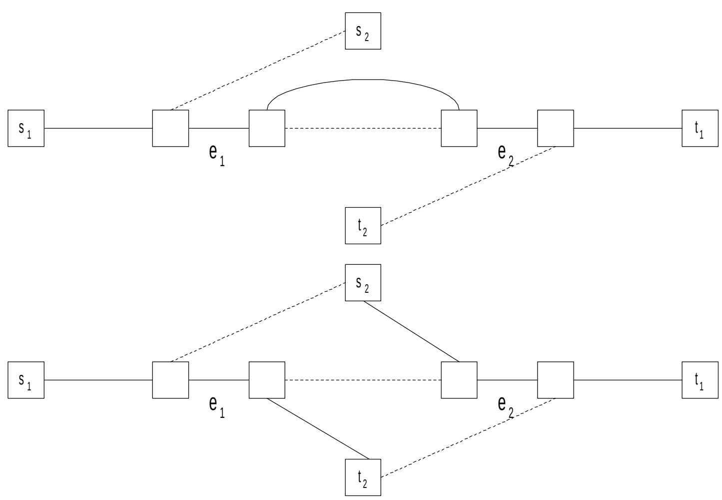
$$\begin{aligned} F_1 &\leq c(s_1, t_1) \\ F_2 &\leq c(s_2, t_2) \\ F_1 + F_2 &\leq c(s_1, s_2; t_1, t_2) \end{aligned} \tag{1.9}$$

and the last inequality is satisfied as an equation when the flows are maximal and the cut is minimal. Also, each of these cut capacities is easily computed.

Theorem 4 *The above conditions are also sufficient.*

Theorem 5 *$\exists$ an optimal solution to the two commodity problem with $F_1 = F_1^*$ where the latter is the optimal value of the first commodity in the absense of the second.*

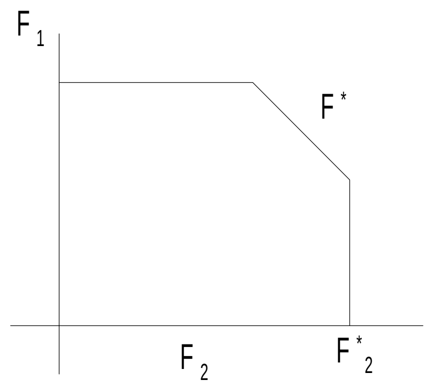
Proof: Let (F_1, F_2) be an optimal solution to the two commodity problem which maximizes F_1. Suppose $F_1 < F_1^*$. If we disregard 2-flows, there is an augmenting path for 1-flow. Traversing this path from s_1 to t_1, let e_1 and e_2 be the first and last edges that are saturated because of 2-flows on them. Since both these edges carry a positive value of 2-flow, either: (i)$\exists$ one path from s_2 to t_2 carrying positive 2-flow using both arcs e_1 and e_2 or (ii)there is one such path using e_1 and another such path using e_2. (See diagrams below).



In either case, we can increase F_1 without changing the value of $F_1 + F_2$. Hence the theorem.□

Corollary 6 *The set of (F_1, F_2) of feasible flows is of the form:*

$$[(F_1, F_2) : 0 \leq F_1 \leq F_1^*;\ 0 \leq F_2 \leq F_2^*; F_1 + F_2 \leq F^*]$$



Given any feasible flow, define the following sets:
$X_1 : s_1 \in X_1$; if $i \in X_1$, and $f^1_{i,j} + \left|f^2_{i,j}\right| < u_{i,j}$, then $j \in X_1$.
$X_2 : s_2 \in X_2$; if $i \in X_2$, and $\left|f^1_{i,j}\right| + f^2_{i,j} < u_{i,j}$, then $j \in X_2$.
$X^b_2 : s_2 \in X^b_2$; if $i \in X^b_2$, and $f^1_{i,j} + f^2_{i,j} < u_{i,j}$, then $j \in X^b_2$.
$X^f_2 : s_2 \in X^f_2$; if $i \in X^f_2$, and $f^1_{j,i} + f^2_{i,j} < u_{i,j}$, then $j \in X^f_2$.

Since $f^1_{i,j} + f^2_{i,j} \leq \left|f^1_{i,j}\right| + f^2_{i,j}$ we may assume without loss that $X_2 \subset X^b_2$. Also, i-flows on the same arc in opposite directions can be cancelled without affecting the total flow; hence we can consider the 1-capacity of an arc (i, j) as follows: 1-capacity in the direction (i, j) is $u^1_{i,j} = u_{i,j} - \left|f^2_{i,j}\right| - f^1_{i,j}$; 1-capacity of the same arc in the direction (j, i) is given by the relation: $u^1_{j,i} = u_{i,j} - \left|f^2_{i,j}\right| + f^1_{i,j}$. Similarly 2-capacities in these directions are respectively: $u^2_{i,j} = u_{i,j} - \left|f^1_{i,j}\right| - f^2_{i,j}$ and $u^2_{j,i} = u_{i,j} - \left|f^1_{i,j}\right| + f^2_{i,j}$. We shall write these capacities as $u^1_{i,j} = u_{i,j} - \left|f^2_{i,j}\right| \pm f^1_{i,j}$ and $u^2_{i,j} = u_{i,j} - \left|f^1_{i,j}\right| \pm f^2_{i,j}$ with the understanding the proper sign will be chosen depending on the direction of the arc we are interested. If $t_1 \in X_1$, we do a regular flow augmentation without altering 2-flows. Similarly if $t_2 \in X_2$ we do a 2-flow augmentation without altering 1-flows. Thus, from now on we may assume without loss that $t_i \notin X_i$ for $i = 1, 2$. If $t_2 \in X^b_2$, then there is a chain P_b, from s_2 to t_2 with the property that for each arc in the chain $u_{i,j} - f^1_{i,j} - f^2_{i,j} > 0$, where the positive direction of the arc agrees with the traversal of the chain from s_2 to t_2. Such a path is called a *backward path* from s_2 to t_2. Similarly, if $t_2 \in X^f_2$, then there is a chain P_f, from s_2 to t_2 with the property that for each arc in the chain $u_{i,j} + f^1_{i,j} - f^2_{i,j} > 0$, where the positive direction of the arc agrees with the traversal of the chain from s_2 to t_2. Such a path is called a *f*orward path from s_2 to t_2. The capacity of such paths is the minimum of these quantities over all the arcs in the path. We say a *double path* exists between s_2 and t_2 if there is one of each kind.

Assume that $t_i \notin X_i$ for $i = 1, 2$. $[f^1_{i,j} + f^2_{i,j} < u_{i,j}; \left|f^1_{i,j}\right| + \left|f^2_{i,j}\right| = u_{i,j}]$ implies that at least one of $f^1_{i,j}$ and $f^2_{i,j}$ is negative. Thus, for each capacitated edge in a backward path at least one of these two quantities is negative. Given that $t_i \notin X_i$, $\exists$ at least one saturated edge on such a backward path. Similarly, $[f_{i,j}^2 - f_{i,j}^1 < u_{i,j}; |f_{i,j}^1| + |f_{i,j}^2| = u_{i,j}]$ implies that we can not have both $f_{i,j}^1 \leq 0$ and $f_{i,j}^2 \geq 0$. Hence, for each capacitated edge on a forward path, either $f_{i,j}^2 < 0$ or $f_{i,j}^1 > 0$ or both.

Definition 2 *A path from s_2 to t_2 is called a* forward path relative to flows $[[f^i, F_i]; i = 1, 2]$ *if it has positive f^1-flows or negative f^2-flows (or both these conditions hold) on each capacitated edge (here positive and negative refer to agreement or disagreement of the direction of flow with the direction of traversal along the path from s_2 to t_2). A path from s_2 to t_2 is called a* backward path relative to flows $[[f^i, F_i]; i = 1, 2]$ *if it has negative f^1-flows or negative f^2-flows (or both of these conditions hold) on each capacitated edge.*

Theorem 7 *Let $[f^1, F_1]$, $[f^2, F_2]$ be a feasible flow with $F_1 = F_1^*$. $F_1 + F_2 < F^* \iff \exists$ a double path from s_2 to t_2. It is this theorem that is the main part of flow augmenting scheme for this problem.*

Suppose $t_i \notin X_i$ for $i = 1, 2$. This implies that we can not increase either commodity without altering the flow of the other. Indeed, we assume that we maximize total 1-flows first and retain at that value; hence we may assume that $F_1 = F_1^*$. Now, using residual capacities $u_{i,j}^r = u_{i,j} - |f_{i,j}^1|$, maximize 2-flows and then we will have such a situation. At this point, if we can find a double path between s_2 and t_2 , then we can increase total 2-flow without changing total 1-flow. New arc flows $\bar{f}$ are given by the relations:

Let

$$g_f^1 = \min_{(i,j) \in P_f : f_{i,j}^1 > 0} [f_{i,j}^1]$$

$$g_b^1 = \min_{(i,j) \in P_b : f_{j,i}^1 > 0} [f_{j,i}^1]$$

$$2\delta_b = \min_{(i,j) \in P_b} [u_{i,j} - f_{i,j}^1 - f_{i,j}^2] > 0$$

$$2\delta_f = \min_{(i,j) \in P_f} [u_{i,j} + f_{i,j}^1 - f_{i,j}^2] > 0$$

and

$$\delta = \min[\delta_b, \delta_f, g_f^1, g_b^1] > 0$$

Now change flows on P_b *and* P_f as follows:

$$\forall\ (i,j) \in P_f : \bar{f}_{i,j}^1 = f_{i,j}^1 - \delta; \bar{f}_{i,j}^2 = f_{i,j}^2 + \delta$$

$$\forall\ (i,j) \in P_b : \bar{f}^1_{i,j} = f^1_{i,j} + \delta; \bar{f}^2_{i,j} = f^2_{i,j} + \delta$$

This leaves the total 1-flow constant (at the value F_1^*) and increases the total 2-flow by 2δ. We need to show that new flow on these arcs is feasible.

$(i,j) \in P_f$: To show that $\left|\bar{f}^1_{i,j}\right| + \left|\bar{f}^2_{i,j}\right| \leq u_{i,j}$ we need to show the following four statements:

$$f^1_{i,j} - \delta + f^2_{i,j} + \delta \leq u_{i,j}$$

$$f^1_{i,j} - \delta - f^2_{i,j} - \delta \leq u_{i,j}$$

$$-f^1_{i,j} + \delta + f^2_{i,j} + \delta \leq u_{i,j}$$

$$-f^1_{i,j} + \delta - f^2_{i,j} - \delta \leq u_{i,j}$$

All except the third of these follow from the fact that the left side is less than or equal to $\left|f^1_{i,j}\right| + \left|f^2_{i,j}\right|$. The third follows from the definition of δ_f and the fact that $\delta \leq \delta_f$.

$(i,j) \in P_b$: To show that $\left|\bar{f}^1_{i,j}\right| + \left|\bar{f}^2_{i,j}\right| \leq u_{i,j}$ we need to show the following four statements:

$$f^1_{i,j} + \delta + f^2_{i,j} + \delta \leq u_{i,j}$$

$$f^1_{i,j} + \delta - f^2_{i,j} - \delta \leq u_{i,j}$$

$$-f^1_{i,j} - \delta + f^2_{i,j} + \delta \leq u_{i,j}$$

$$-f^1_{i,j} - \delta - f^2_{i,j} - \delta \leq u_{i,j}$$

All except the first of these follow from the fact that the left side $\leq \left|f^1_{i,j}\right| + \left|f^2_{i,j}\right|$. The first follows from the definition of δ_b and the fact that $\delta \leq \delta_b$. *Note that if an arc is used in both these paths in opposite directions,* 1-*flow increases by* 2δ *and* 2-*flow changes cancel each other. However, no cpacity will be exceeded because of the definition of* δ. *The same arguemnt applies if the arc is used in both paths in the same direction.* Hence, the existence of a double path when $t_i \notin X_i$, for $i = 1, 2$ gives an augmentation scheme. *Once this augmentation is done, we may need to do some* 2-*flow augmentations without altering* 1-*flows.* We use double path augmentation only when we are unable to augment 2-flow without altering 1-flow.

To show the converse we need to study the nature of the sets X_2^b and X_2^f in greater detail. Suppose there is a pair of nodes $(i,j) \ni u_{i,j} - f_{i,j}^1 - f_{i,j}^2 = 0$. Since $u_{i,j} \geq \left|f_{i,j}^1\right| + \left|f_{i,j}^2\right| \geq f_{i,j}^1 + f_{i,j}^2 = u_{i,j}$ in such a case we would have $f_{i,j}^1 = \left|f_{i,j}^1\right| \geq 0$ and $f_{i,j}^2 = \left|f_{i,j}^2\right| \geq 0$. If for any partition of the nodes into $(S, \bar{S})$ with $u_{i,j} - f_{i,j}^1 - f_{i,j}^2 = 0 \ \forall \ i \in S, j \in \bar{S}$, we have $f_{i,j}^k \geq 0 \ \forall \ k$, and $i \in S$ and $j \in \bar{S}$. In this case, if s_k and t_k are not separated by the cut $(S, \bar{S})$, then $f_{i,j}^k = 0 \ \forall \ i \in S, j \in \bar{S}$. This is true because the net flow across such a cut for this commodity is zero and all arcs have nonnegative flows for this commodity.

If $\exists$ no backward path then $(X_2^b, \bar{X}_2^b)$ is such a cut and it separates s_2 from t_2. Hence, if s_1 and t_1 are not separated by this cut, then $f_{i,j}^1 = 0 \ \forall \ i \in X_2^b, j \in \bar{X}_2^b$. Thus, $f_{i,j}^2 = u_{i,j}$ for all these arcs and hence $F_2 = F_2^* = u(X_2^b, \bar{X}_2^b)$ and we have reached optimalty. If s_1 and t_1 are separated by this cut, then since $F_1^* > 0$, there are arcs with $f_{i,j} > 0$ across this cut. In this case, we have $F_1 + F_2 = u(X_2^b, \bar{X}_2^b)$ and hence we have optimality. Thus, if $\exists$ no backward path we have optimality. The proof that we have optimality if $\exists$ no forward path is similar. Hence the theorem.□

Algorithm H:

Step 0: Find $[f^1, F_1^*]$ using $u_{i,j}$ and a single commodity algorithm.
Step 1: Find $[f^2, F_2]$ using as arc capacities $u_{i,j} - \left|f_{i,j}^1\right|$.
Step 2: Try to locate double paths from s_2 to t_2. If they do not exist, the current flow is optimal. If they are found calculate δ and do the flow alterations as described above. Go to step 1.

We have already shown that steps 0 — 2 are valid. *The most important part in proving that this algorithm is finite, for even integral data, is that the solution is always integral.*

Step 0 clearly produces *even* integers for f^1 and F_1 and step 1 does the same for f^2 and F_2. δ is given by the relations:

$$g_f^1 = \min_{(i,j) \in P_f : f_{i,j}^1 > 0} [f_{i,j}^1]$$

$$g_b^1 = \min_{(i,j) \in P_b : f_{j,i}^1 > 0} [f_{j,i}^1]$$

$$2\delta_b = \min_{(i,j) \in P_b} [u_{i,j} - f_{i,j}^1 - f_{i,j}^2] > 0$$

$$2\delta_f = \min_{(i,j) \in P_f} [u_{i,j} + f_{i,j}^1 - f_{i,j}^2] > 0$$

and

$$\delta = \min[\delta_b, \delta_f, g_f^1, g_b^1] > 0$$

If integrality is not maintained then there is some point in the algorithm when this happens for the first time. At this point, δ must be fractional (the previous δ is integral) and this can happen only if $\min[\delta_b, \delta_f]$ is also fractional. This in turn implies $2\min[\delta_b, \delta_f]$ is odd.

Let (i,j) be an arc with $\bar{f}^1_{i,j} = f^1_{i,j} + \delta$ and $\bar{f}^2_{i,j} = f^2_{i,j} - \delta$.

$u_{i,j} \pm \left|f^1_{i,j} + \delta\right| \pm \left|f^2_{i,j} - \delta\right|$ is even if $u_{i,j} \pm \left|f^1_{i,j}\right| \pm \left|f^2_{i,j}\right|$ is even and δ is integral. Thus, starting with integral δ we can never have nonintegral δ and this completes the proof of integrality and finiteness.□

We know that single commodity algorithms can be made finite (and indeed polynomial) even in the presense of irrational data using real arithmetic. This can be done here too and this is shown below. But before we pursue that line, we show that this sort of result does not hold for more commodities. See example in Hu pp. 188 or Rothfarb and Frisch pp. 47 with 3 commodities. Even the simple case of 4 commodities with only sources and sinks for each as the nodes of a complete graph on them is a counter example for maxflow mincut.

Sakarovitch's Algorithm:

M. Sakarovitch formualtes the problem slightly differently as follows. He assumes that we are given a *directed* graph $G = [N, A]$ with the property $(i,j) \in A \iff (j,i) \notin A$; such a graph is said to be *antisymmetric*. Essentially he has arbitrarily assigned direction to the edges. Also, let $u_{i,j}$ be the capacity of the arc; negative arc flows are permitted (these mean that the direction of flow is against the chosen direction for the arc). Hence the problem is:

$$\sum_j f^k_{i,j} - \sum_j f^k_{j,i} = \begin{array}{rl} F_k & i = s_k \\ 0 & i \neq s_k, t_k \\ -F_k & i = t_k \end{array}$$

$$\sum_k \left|f^k_{i,j}\right| \leq u_{i,j} \forall\ (i,j) \in A$$

$$\max \sum_k |F_k|$$

Remarks:

1. Sources and sinks can be interchanged. Hence the objective function can be replaced by $\sum F_k$.

2. A network is *eulerian* if $u_{i,j}$ are integral and $\sum_j u_{i,j}$ is even $\forall\ i$.

3. We can rewrite the capacity constraints as:

$$-u_{i,j} \leq f^1_{i,j} + f^2_{i,j} \leq u_{i,j} \forall (i,j)$$

$$-u_{i,j} \leq f^1_{i,j} - f^2_{i,j} \leq u_{i,j} \forall (i,j)$$

4. There is a redundant constraint in each set of conservation equations which can be dropped.

This suggests the following transformation: Let

$$g^1_{i,j} = (f^1_{i,j} + f^2_{i,j})/2$$

$$g^2_{i,j} = (f^1_{i,j} - f^2_{i,j})/2$$

Then,

$$f^1_{i,j} = (g^1_{i,j} + g^2_{i,j})/2$$
$$f^2_{i,j} = (g^1_{i,j} - g^2_{i,j})/2$$

If we now let

$$h^1_{i,j} = g^1_{i,j} + u_{i,j}/2$$
$$h^2_{i,j} = g^2_{i,j} + u_{i,j}/2$$

we get the equations:

$$h^1_{i,j} = (u_{i,j} + f^1_{i,j} + f^2_{i,j})/2$$

$$h^2_{i,j} = (u_{i,j} + f^1_{i,j} - f^2_{i,j})/2$$

and

$$f^1_{i,j} = h^1_{i,j} + h^2_{i,j} - u_{i,j}$$
$$f^2_{i,j} = h^1_{i,j} - h^2_{i,j}$$

Using the relationship between f and g we have the following reformulation:

Problem I:

$$E(g^1 + g^2) + R^1 F_1 = 0$$

$$E(g^1 - g^2) + R^2 F_2 = 0$$

$$-u/2 \leq g^1 \leq u/2$$

$$-u/2 \leq g^2 \leq u/2$$

$$\max F_1 + F_2$$

Here E is the node-arc incidence matrix (it is possible to assume without loss that the row corresponding to s_1 is removed in both commodity conservation equations) and R^1 is the unit vector with 1 in the position corresponding to t_1. R^2 is a vector all of whose components are zero except in row corresponding to s_2 where there is a $+1$ and row correesponding to t_2 where there is a -1. Simplifying the first two equations we get:

Problem II:

$$Eg^1 \ + [R^1F_1 + R^2F_2]/2 = 0$$

$$Eg^2 \ + [R^1F_1 - R^2F_2]/2 = 0$$

$$-u/2 \leq g^1 \leq u/2$$

$$-u/2 \leq g^2 \leq u/2$$

$$\max F_1 + F_2$$

The dual of the above LP is:

Problem II':

$$\pi^1 E + \beta^1 - \gamma^1 = 0$$

$$\pi^2 E + \beta^2 - \gamma^2 = 0$$

$$\pi^1_{t_1} + \pi^2_{t_1} = 2$$

$$[\pi^1_{t_2} - \pi^1_{s_2}] + [\pi^2_{t_2} - \pi^2_{s_2}] = 2$$

$$\beta^1; \beta^2; \gamma^1; \gamma^2 \geq 0$$

$$\min[\beta^1 + \gamma^1 + \beta^2 + \gamma^2]u/2$$

The asymmetry in these two sets of equations is due to the dropping of the same constraint in both which is redundant.

Using h, we get the primal problem to be:

Problem III:

$$Eh^1 \ + [R^1F_1 + R^2F_2]/2 = Eu/2$$

$$Eh^2 \ + [R^1F_1 - R^2F_2]/2 = Eu/2$$

$$0 \leq h^1 \leq u; 0 \leq h^2 \leq u$$

$$\max F_1 + F_2$$

The dual of the above LP is:

Problem III':

$$\pi^1 E + \beta^1 \geq 0$$

$$\pi^2 E + \beta^2 \geq 0$$

$$\pi^1_{t_1} + \pi^2_{t_1} = 2$$

$$[\pi^1_{t_2} - \pi^1_{s_2}] + [\pi^2_{t_2} - \pi^2_{s_2}] = 2$$

$$\beta^1 \geq 0; \beta^2 \geq 0$$

$$\min[\beta^1 + \beta^2]u + [\pi^1 + \pi^2]Eu/2$$

Lemma 8 *If $(\pi^1, \pi^2, \beta^1, \beta^2, \gamma^1, \gamma^2)$ is optimal to II', then $(\pi^1, \pi^2, \beta^1, \beta^2)$ is optimal to III'. Conversely, if $(\pi^1, \pi^2, \beta^1, \beta^2)$ is optimal to III', then $(\pi^1, \pi^2, \beta^1, \beta^2, \gamma^1, \gamma^2)$ is optimal to II' where $\gamma^1 = \pi^1 E + \beta^1$ and $\gamma^2 = \pi^2 E + \beta^2$.*

Remark: Let (h^1, h^2, F_1, F_2) be a solution to Problem III. Then $((u - h^1), (u - h^2), -F_1, -F_2)$ is also a solution to III whose value for $|F_1| + |F_2|$ is the same. This would correspond to interchanging the sources and sinks on both commodities. *Of course, you can do this sort of thing only on one of the commodities.* When this is done for both commodities, we shall call the resulting problem $I\hat{I}I$.

The feasible region for each arc is depicted in the following diagram:

Lemma 9 *Let $[F_1, F_2]$ be an optimal solution to problems I – III. Let $\pi^1, \pi^2, \beta^1, \beta^2, \gamma^1, \gamma^2$ be an optimal solution to II' and let $C^k = [(i,j) : \beta^k_{i,j} + \gamma^k_{i,j} > 0]$ for $k = 1, 2$. Let $C = \cup_k C^k$. Then, C contains a cut.*

Proof: (i)Let P^k be a path from s_k to t_k for $k = 1, 2$. Suppose $P^1 \cap C = \phi$. From II' we have:

$$\pi^k_j - \pi^k_i = \beta^k_{i,j} - \gamma^k_{i,j} \; \forall \, (i,j) \in A \; and \; \forall \; k$$

and hence $\forall \, (i,j) \in P^1$, $\pi^k_j - \pi^k_i = 0 \; \forall \; k$ under this hypothesis. This implies $\pi^k_{t_1} = 0 \; \forall \; k$ and this contradicts $\pi^1_{t_1} + \pi^2_{t_1} = 2$. If $P^2 \cap C = \phi, (i,j) \in P^2$ implies

$$\pi^k_j - \pi^k_i = 0 \; (i,j) \in P^2 \; and \; \forall \; k$$

which in turn implies that $\pi^k_{t_2} - \pi^k_{s_2} = 0 \; \forall \; k$ and this is a contradiction to the constraint

$$\pi^1_{t_2} - \pi^1_{s_2} + \pi^2_{t_2} - \pi^2_{s_2} = 2$$

Hence the lemma.□

Lemma 10 *Every optimal basis to III is triangular and hence has a determinant equal ± 1.*

Proof: Clearly, we can perform a *reduction* of a basis matrix B if there is only one nonzero element in some row or column of the matrix; the reduced matrix is triangular iff the original one is. Perform the following reductions on an optimal basis of Problem III: $h^k_{i,j}$ if its slack is not in the basis (in this case $h^k_{i,j} = u_{i,j}$). After all of these are done B is of the form:

$$B = \begin{matrix} E^1 & 0 & R^1 & R^2 \\ 0 & E^2 & R^1 & -R^2 \end{matrix}$$

A column in E^k corresponds to a variable $h^k_{i,j}$ that satisfies the relation: $0 < h^k_{i,j} < u_{i,j}$ and by LP duality implies $\beta^k_{i,j} = 0$; and $\pi^k_i - \pi^k_j + \beta^k_{i,j} = 0$ ($= \gamma^k_{i,j}$ of II'). Hence, columns of E^k correspond to variables $\beta^k_{i,j} = \gamma^k_{i,j} = 0$. Let $A^k = [(i,j) : \beta^k_{i,j} = \gamma^k_{i,j} = 0]$. It follows from lemma 2 that each of $G^k = [N, A^k]$ has at least two components none of which contains both source and sink of either commodity. Let H be the connected component of G^2 that contains one of s_2 or t_2 but not t_1. Without loss, assume that H contains t_2. Since $\det B \neq 0$, H does not contain a cycle. Hence $\exists$ a tip vertex $\neq t_2$ in H. The corresponding row of B has a single nonzero element in B. By successive reductions of B (and corresponding operations on H) H is reduced to vertex t_2. Now a reduction on the row of vertex t_2 and the last column is performed. Note that neither R^1 nor E^2 (at this point) has any nonzeroes in this row. At this stage we are left with a matrix of the form:

$$\hat{B} = \begin{matrix} E^1 & 0 & R^1 \\ 0 & \hat{E}^2 & R^1 \end{matrix}$$

Which is clearly totally unimodular. *Note that B is not necessarily totally unimodular.* Using similar arguements we can show that optimal bases of II are also unimodular□

Theorem 11 *$\exists$ a cut C satisfying the relation: $F_1^* + F_2^* = u(C)$ where $u(C) = \sum_{(i,j)\in C} u_{i,j}$.*

Proof: Let $\pi^1, \pi^2, \beta^1, \beta^2, \gamma^1, \gamma^2$ be an optimal solution to II'. From lemma 3 and its corresponding result for II', and the fact that right side of II' is even, we deduce that β^k and γ^k are even. Now using the fact that $\beta^k_{i,j}$ measures the increase in the total flow for a unit increase in $u_{i,j}/2$, we have $\beta^k_{i,j} \leq 2$ and hence 0 or 2. (*You can also prove this directly*). Similarly $\gamma^k_{i,j}$ is also 0 or 2. Now let C^k be definied as in lemma 2. Hence, by LP duality we have:

$$\begin{aligned} F_1^* + F_2^* &= \sum_{(i,j)\in C^1} u_{i,j} + \sum_{(i,j)\in C^2} u_{i,j} \\ &= \sum_{(i,j)\in C} u_{i,j} + \sum_{(i,j)\in C^1 \cap C^2} u_{i,j} \end{aligned}$$

$$\geq \sum_{(i,j)\in C} u_{i,j} \qquad (1.10)$$

$F_1^* + F_2^* = \sum_{(i,j)\in C^1} u_{i,j} + \sum_{(i,j)\in C^2} u_{i,j}$
$= \sum_{(i,j)\in C} u_{i,j} + \sum_{(i,j)\in C^1 \cap C^2} u_{i,j}$
$\geq \sum_{(i,j)\in C} u_{i,j}$
Hence equality holds in the last step and the second term in the second step is zero. This proves the theorem. *Note that this proof did not require integrality of capacities. This is how it should be; unfortunately T.C. Hu's proof needed integrality.*

Theorem 12 *∃ an integer optimal solution if u is even.*

Theorem 13 *∃ an integer optimal solution if the network is eulerian.*

The first of these follows from lemma 3 applied to II and the second from lemma 3 applied to III. Now we describe Sakarovitch's algorithm for this problem which differs from Hu's. $[S]$ solves the problem by solving *two single commodity problems*. The advantage that this has is that we can now resolve the case of nonintegral capacities. Make two copies $G^1 = [N^1, A^1]$ and $G^2 = [N^2, A^2]$ of G. Note that N, N^1, and N^2 are identical clones and so are the corresponding arcs. We will also use this notation for subsets. We define lower and upper bounds on arcs of G^1 and G^2 as follows:

$$l^1_{i,j} = l^2_{i,j} = -u_{i,j}/2; u^1_{i,j} = u^2_{i,j} = u_{i,j}/2$$

The *capacity* of a cut $(S^1, \bar{S}^1)$ separating s^1_1 (the clone of s_1 in G) from t^1_1 (the clone of t_1 in G) is given by:

$$\sum_{\substack{i\in S^1\\ j\notin S^1}} u^1_{i,j} - \sum_{\substack{i\in S^1\\ j\notin S^1}} l^1_{j,i} = u^1(S^1, \bar{S}^1) - l^1(\bar{S}^1, S^1)$$

Such a cut is said to be *saturated* if all arcs of the first term are at upper bound and all arcs of the second are at lower bound.

Algorithm $[S]$:

Step 1: Maximize flow between t^1_1 and s^2_1 using the network $H = [N^1 \cup N^2, A^1 \cup A^2 \cup (t^1_1, s^2_1) \cup (t^2_1, s^1_1)]$ in which the bounds on the last two arcs are $(-\infty, \infty)$. Let $[\varphi, \Phi_1]$ be the resulting optimal flow.
Step 2: Maximize flow between t^1_2 and t^2_2 using the network K which has the same nodes as H and all arcs of H and two additional arcs: (t^1_2, t^2_2) and (s^2_2, s^1_2). Now each of the arcs $(t^1_1, s^2_1) \cup (t^2_1, s^1_1)$ have lower bound equal to upper bound equal to $\Phi_1 = \varphi(t^1_1, s^2_1) = \varphi(t^2_1, s^1_1)$. Use φ as the starting flow. Let the resulting flow be $[\psi, \Psi]$.

Theorem 14 *Optimal solution to the first problem is given by:*

$$f^{*1}_{i,j} = \psi_{i^1,j^1} + \psi_{i^2,j^2}$$

$$f^{*2}_{i,j} = \psi_{i^1,j^1} - \psi_{i^2,j^2}$$
$$F_1^* = 2\psi_{t_1^1,s_1^2} = 2\psi_{t_1^2,s_1^1}$$
$$F_2^* = 2\psi_{t_2^1,t_2^2}$$

Proof: It is easy to see that flow conservation for each commodity is satisfied for each intermediate node. For 1-flow at s_1,

$$\sum f^{*1}_{s_{1},,j} - \sum f^{*1}_{j,s_1} = \psi(t_1^2, s_1^1) + \psi(t_1^1, s_1^2) = F_1^*$$

Those of both flows for each of these nodes is shown in a similar manner. To show that these satisfy capacity constraints we note:

$$\left|f^{*1}_{i,j}\right| + \left|f^{*2}_{i,j}\right| \leq 2 \ \max[\left|\psi_{i_1,j_1}\right|, \left|\psi_{i_2,j_2}\right|] \leq u_{i,j}$$

To prove optimality need some preliminary results which are taken up now. At the end of step 1 of the algorithm, $\exists$ is a saturated cut $(Y^1, \bar{Y}^1)$ separating s_1^1 from t_1^1 in G^1 and a cut $(Y^2, \bar{Y}^2)$ separating s_1^2 from t_1^2 in G^2. Note that both these cuts correspond to a cut $(Y, \bar{Y})$ in G. At the end of step 2, $\exists$ either a saturated cut $(Z^1, \bar{Z}^1)$ separating s_2^1 form t_2^1 in G^1 or a saturated cut $(Z^2, \bar{Z}^2)$ separating t_2^2 form s_2^2 in G^2 or both.

Lemma 15 $(i)(Y^1, \bar{Y}^1) \cap (Z^1, \bar{Z}^1) \neq \phi \Longrightarrow (Z^1, \bar{Z}^1)$ *separates* s_1^1 *from* t_1^1*;* $(ii)(Y^2, \bar{Y}^2) \cap (Z^2, \bar{Z}^2) \neq \phi \Longrightarrow (Z^2, \bar{Z}^2)$ *separates* s_1^2 *from* t_1^2*.*

Proof: We will show (i); (ii) is proved along similar lines. Suppose both s_1^1 and $t_1^1 \in \bar{Z}^1$; let $(Y^1, \bar{Y}^1) \cap (Z^1, \bar{Z}^1) = M^1 \neq \phi$; $P^1 = (Y^1, \bar{Y}^1) - M^1$; $Q^1 = (Z^1, \bar{Z}^1) - M^1$. (See figure below)

Let M, P, Q be the corresponding sets in G.

$$\psi_{t_1^1,s_1^1} = \psi_{t_1^2,s_1^1} = [u(M) + u(P)]/2$$

$$\psi_{t_2^1,t_2^2} = \psi_{s_2^1,s_2^2} = [u(M) + u(Q)]/2$$

The first of these equations is forced and the second follows from the fact that there is a cut with only these four arcs. In G^2, $P^2 \cup Q^2$ correspond to a cut separating $\{s_1^2, t_2^2\}$ from $\{s_2^2, t_1^2\}$. The capacity of this cut equals $[u(P) + u(Q)]/2$. Thus we have:

$$\begin{aligned} 2[\psi_{t_1^1,s_1^2} + \psi_{t_2^1,t_2^2}] &= u(P) + u(Q) + 2u(M) \\ &\leq u(P) + u(Q) \end{aligned} \quad (1.11)$$

$2[\psi_{t_1^1,s_1^2} + \psi_{t_2^1,t_2^2}] = u(P) + u(Q) + 2u(M)$
$\leq u(P) + u(Q)$

which implies $u(M) = 0$ and hence $M = \phi$ and this is a contradiction. The same proof applies if s_1^1 and $t_1^1 \in Z^1$.□

Now we return to the proof of the theorem 4.

Proof of Theorem 4: (contd) To show optimality of the solutions we will show that $\exists$ a saturated cut C in G. There are two cases:

(a) $s_1^1 \in Z^1, t_1^1 \notin Z^1$ (or $s_1^2 \in Z^2, t_1^2 \notin Z^2$) :Let $C = (Z, \bar{Z})$ where Z corresponds to Z^1 (or Z^2).

$$\begin{aligned} 2[\psi_{t_1^1,s_1^2} + \psi_{t_2^1,t_2^2}] &= 2[u^1(Z^1,\bar{Z}^1) - l^1(\bar{Z}^1, Z^1)] \\ &= F_1 + F_2 \qquad (1.12) \\ &= u(C) \qquad (1.13) \end{aligned}$$

$$2[\psi_{t_1^1,s_1^2} + \psi_{t_2^1,t_2^2}] = 2[u^1(Z^1,\bar{Z}^1) - l^1(\bar{Z}^1, Z^1)] = F_1 + F_2 = u(C).$$

(b) One of $\{[s_1^1$, and $t_1^1 \in Z^1]$, $[s_1^1$, and $t_1^1 \in \bar{Z}^1]\}$ holds (or one of $\{[s_1^2$, and $t_1^2 \in Z^2]$, $[s_1^2$, and $t_1^2 \in \bar{Z}^2]\}$ holds). For the sake of specificity, assume $[s_1^1$, and $t_1^1 \in Z^1]$. Let $C = (Y,\bar{Y}) \cup (Z,\bar{Z})$ where Y corresponds to Y^1 and Z corresponds to Z^1. From previous lemma we have: $(Y,\bar{Y}) \cup (Z,\bar{Z}) = \phi$. More over, in this case,

$$\psi_{s_2^2,s_2^1} = \psi_{t_2^1,t_2^2} = u^1(Z,\bar{Z}) - l^1(\bar{Z}, Z)$$

Also, either s_2^1, and t_2^1 are both in Y^1 or are both in $\bar{Y}^1$; if not, then $(Y^1,\bar{Y}^1)$ would have been a saturated cut separating t_2^2 from s_2^2 at the beginning of step 2. Thus,

$$\psi_{t_1^2,s_1^1} = \psi_{t_1^1,s_1^2} = u^1(Y^1,\bar{Y}^1) - l^1(\bar{Y}^1, Y^1)$$

$$\begin{aligned} F_1 + F_2 &= 2[\psi_{t_1^2,s_1^1} + \psi_{s_2^2,s_2^1}] \\ &= 2[u^1(Z,\bar{Z}) - l^1(\bar{Z}, Z)] + 2[u^1(Y^1,\bar{Y}^1) - l^1(\bar{Y}^1, Y^1)] \\ &= u(C) \qquad (1.14) \end{aligned}$$

$F_1 + F_2 = 2[\psi_{t_1^2,s_1^1} + \psi_{s_2^2,s_2^1}] = 2[u^1(Z,\bar{Z}) - l^1(\bar{Z}, Z)] + 2[u^1(Y^1,\bar{Y}^1) - l^1(\bar{Y}^1, Y^1)] = u(C)$.

This proves the theorem.□

Corollary 16 $\exists$ *an optimal solution with* $F_1 = F_1^*$.

2
General Multicommodity Flows

In this section we consider maximum multicommodity flow problem on undirected networks. Our approach is partially inspired by the works of Karzanov, Lomonosov et al.

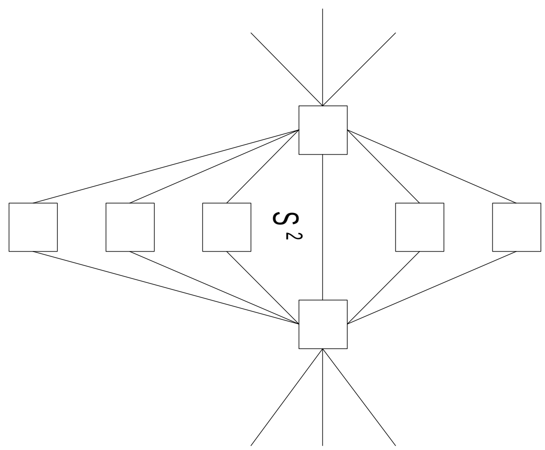
We are given two undirected graphs $G = [N, E, c]$ and $H = [T, U]$ with $T \subseteq N$. The network H is called the *commodity graph;* T are called the *terminals,* the elements of U denote origin-destination pairs for these commodities and c denotes nonnegative edge capacities. We wish to maximize the sum of all these commodity flows. We assume throughout this work that H has no isolated nodes.

Example 1 $|U| = 1$*: This is the well known single commodity case. We have the famous max-flow-min-cut theorem. Here if c is integral, then we can be assured that there are integral optimal solutions.*

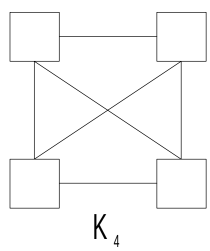
Example 2 *The case when* H *is a star network (i.e.* $U = \{(1, j) : j \neq 1\}$*) can be easily converted to the above case. This case is denoted by* S*. This also assures integrality property for this case.*

Example 3 $H = K_2$*: This is the well known Two Commodity Flow Problem solved first by T.C. Hu. Here we still have max-flow-min-cut equality but the integrality property is no longer valid. Instead, if c is integral we get half integral optimal solutions. This was generalized by Rothschild and Whinston to the Eulerian case for which integrality holds.*

Example 4 *The case where* $H = S^2$ *which is the union of two star graphs can be reduced to the case* K_2 *by the same reduction that reduces* S *to* *single commodity case. Remarks regarding integrality or half integrality in the above also apply here.*



The next important case was solved by Kuperschtoch and here $H = K_n$ for some n.

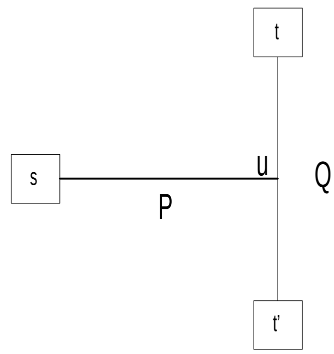


Let F_s^* be the maximum total flow on G for the star problem $H_s = [T, U_s]$ where $U_s = \{(s,j) : j \in T - s\}$. Clearly F_s^* equals the value of the minimum cut separating s from $T - s$.

Theorem 17 *There is an optimal solution to the problems with* $H = K_n$ *in which the total flow of commodities* $\{(s,j); j \in T - s\}$ *equals* F_s^**. This implies, that the maximum total flow* F *for* H *is* $\frac{1}{2}\sum_s F_s^*$*. Moreover, there is a half integral flow if* c *is integral (and an integral flow if it is eulerian).*

Proof: Suppose in some optimal solution to the problem, $F_s < F_s^*$. This means that if we ignore the flows of all other commodities, then there is an augmenting path P from s to some $t \in T - s$. But this can not be augmenting path in the original problem if we do not change any of the remaining commodity flows. This implies that some edges along this path are saturated due to the presence of other commodity flows; let $e = (u, v)$ be the *first* such edge (traversed in this order) along P. This implies there is some path Q passing through u connecting two terminals $t, t' \in T - s$

(as shown below) carrying positive flow. Please note that the arcs on P in the diagram below are either unsaturated or have flow of the type (s, j) in the reverse direction. This is similar to the single commodity flow labeling.



Reducing flow of this commodity on Q by $\delta > 0$, will allow us to increase flow on each of the two paths $P[s,u] \cup Q[u,t]$ and $P[s,u] \cup Q[u,t']$ by δ which yields a net increase of δ for the total flow. Hence if the original flow was optimal we can not have this situation. Hence the first part of the theorem. In any feasible solution, we have:

$$\sum_{\substack{u=(s,j)\\ j\in T-s}} F_u \leq F_s^* \forall s \in T$$

Adding all these equations we get the result that the total flow is less than or equal to $\frac{1}{2}\sum_{s\in T} F_s^*$. That equality is achieved at optimality follows from the above discussion. This proves the second part of the theorem. The third part is done by an algorithm due to Cherkasskii called the T-routine. Under the assumption of integrality for c the algorithm is shown to maintain half integrality at all times and hence the last part of the theorem.

We describe this algorithm now. To prove a slightly stronger result, we assume that c satisfies the inner Eulerian condition described below and provide integral solutions:

Definition 3 *We say that c is inner Eulerian with respect to a given $G = [N, E]$ and $H = [T, U]$ if c is integral and $\sum_j c_{i,j}$ is even for all $i \in N - T$.*

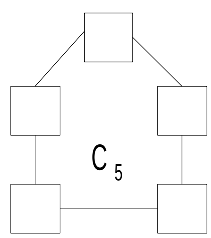
2.0.1 T-Routine:

Throughout this routine we assume integral flows (starting with zero flows). If we consider flows commodities for which node s is a terminal, and keep all other commodity flows frozen (at integral values), the residual capacities of arcs satisfy inner Eulerian condition since original capacities did and each *line* of flow of all commodities reduce the total capacity by an even integer at each inner node. Hence, there is a double path of unsaturated arcs to any node that has such a path at all. This allows us to choose a δ in the above discussion to be integral and hence the new improved flow is also integral. This concludes the discussion on this theorem.

Remark 1 *Note that no claim is made at this point regarding the complexity of the algorithm other than that it is finite.*

Example 5 *If $H \subseteq K_4$, then it is easy to show that either: (i)$H = K_4$ in which case the T-routine solves the maximum flow problem or (ii)we can convert the problem to the case where $|U| = 2$ by the same transformation that converts $H = S$ to a single commodity problem. Thus, we need not consider these cases. Moreover, integrality is assured in these cases as well when the capacities are Eulerian.*

Example 6 *The only case with five termini that can not be converted to the previous cases (by the transformation that turns a star into a single commodity) and still determined by cut capacities is $H = C_5$; a five cycle.*



C_5 is solved by Karzanov-Lomonosov by what Lomonosov calls the drain transformation which solves the so called Locking problem and this in turn solves the maxflow problem. It turns out that integrality under Eulerian condition is false in this case.

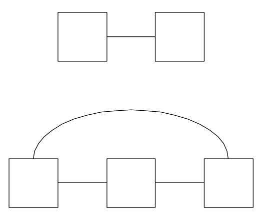
For $i, j \in T$ let $C^*_{i,j}$ represent the maximum total flow with one terminal in the set $\{i, j\}$ and the other in the set $T - \{i, j\}$. Clearly this value is determined by the capacity of some cut (since this is the case of multiple origin mutliple destinations case but with single commodity). We are only interested in these values for $(i, j) \notin C_5$. For example, if the arcs of C_5 are
$\{(i, i+1) : 1 \leq i \leq 5 \; mod(5)\}$ then we would be interested in $F^*_{i,j}$ for those cases where $j = i + 2 \; mod(5)$.

Lemma 18 *The following relations are satisfied by any feasible flow:*

$$
\begin{aligned}
F_{1,2} + F_{2,3} + F_{3,4} + F_{5,1} &\leq C^*_{1,3} \\
F_{4,5} + F_{2,3} + F_{3,4} + F_{5,1} &\leq C^*_{3,5} \\
F_{1,2} + F_{2,3} + F_{4,5} + F_{5,1} &\leq C^*_{5,2} \\
F_{1,2} + F_{2,3} + F_{3,4} + F_{4,5} &\leq C^*_{2,4} \\
F_{1,2} + F_{4,5} + F_{3,4} + F_{5,1} &\leq C^*_{4,1}
\end{aligned}
$$

Adding all these we get an upper bound on the total flow equal to $\frac{1}{4}\sum_{i=1}^{5} C^*_{i,i+2mod(5)}$. We will show that this bound is achieved. Further, this value is half integral if c is Eulerian since each cut is even. The case with $H = C_5$ and $G = \bar{H}$ and $c \equiv 1$ shows that even when G is Eulerian, there may only be half integral flows. In this the total flow at optimality is $2\frac{1}{2}$. Karzanov has shown the stronger fact that the fractionality index for this problem is 4.

Example 7 $H = K_2 \cup K_3$:



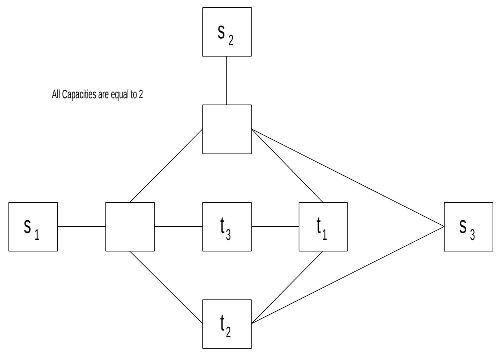
In all the above cases max-flow is determined by cut capacities. Now we move on to cases involving $|T| = 5$ aptly called five terminus flows by Karzanov. All the remaining cases where $H \subseteq K_5$ and not convertible to the previous cases can be shown to reduce to the case where $H = K_2 \cup K_3$. We take up this case now. It turns out that integrality under Eulerian condition is still true; but maxflow is not determined by cut capacities. Karzanov showed that we need what he calls $(2,3)-$metric and that these are also sufficient in this case.

For the purposes of integrality of the maxflow and feasibility problem it has been shown that the absence of some special subgraphs in H is important. For example, the absence of $K_2 \cup K_2 \cup K_2$ as a subgraph of H yields a subgraph of one of K_4, C_5, or S^2. This is important for the feasibility problem. Absence of either $K_2 \cup K_2 \cup K_2$ or $K_2 \cup K_3$ yields a subgraph of K_5, S^2, ot $K_3 \cup K_3$. This is important for cut dependent problems. Nice algorithms exist for the maximinum flow problems for all these cases except $K_3 \cup K_3$.

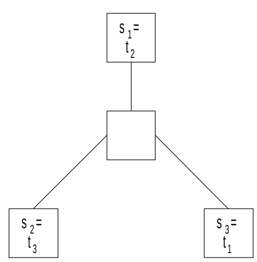
2.1 Examples in Multicommodity Flows

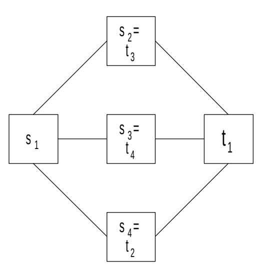
Example I:

$G = [N, E]$; $N = \{s_k, t_k, k = 1, 2, 3\}$; $E = \{(s_2, x), (s_3, y), (t_1, t_3), (t_1, y), (s_1, x), (s_1, t_2), (x, t_3), (x, y), (t_2, t_3), (t_2, y)\}$; $u_{i,j} = 2 \; \forall \; (i, j) \in E$.



For this example with $k = 3$, the inequalities $u(X, \bar{X}) \geq \sum F_k$ where the sum is over those commodities whose s_i and t_i are separated by this cut is not sufficient to achieve these flow values. For example, in this problem take $F_1 = 4$; $F_2 = 2$; and $F_3 = 1$. Any two commodities is achievable and min cut separating all three is 8. Yet these values are not achievable. **T.C.Hu** gives an example (by contracting arcs (x, s_2) and (y, s_3) in which the above values are not achievable but max flow equals min cut. In the example given, even this is not true. See "A Two commodity Cut Theorem" by **P.D. Seymour**, Disc. math (1978), **23**, 177-181 for a simpler example of this kind.





Rothfarb and Frisch show that if every node is either an origin or a destination, max flow equals min cut for $k = 3$. For $k = 4$ even this is not true. This property is now termed as Q^{+}-MFMC property. The problem of determining when maxflow-mincut holds is still an open question.

2.2 Multicommodity Flows References

2.2.1 Initial results

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