1
Blocking Systems

1.1 Examples

I Given a (directed) network $G = [N, A]$ and *capacities* $c_{i,j}$ on arc (i, j), $\forall\ (i, j) \in A$, find a (directed) path P^* from a specifed node s to another specified node $t \ni \min_{(i,j)\in P^*} c_{i,j} = \max_P \min_{(i,j)\in P} c_{i,j}$. Such a problem is known as an instance of the *bottleneck problem.*

II Another instance is called the *bottleneck assignment problem* in which the minimum is over the cells selected in an assignment and the maximum is over all feasible assignments. This problem is useful in the context of assembly lines.

We now discuss the structure of these problems and this is the area of *blocking systems.*

Definition 1 *A family $\mathcal{P}$, of subsets of a finite set E that satisfies the relation: $P \in \mathcal{P}$, $Q \subset \mathcal{P} \Longrightarrow Q \notin \mathcal{P}$, is called* a clutter *on E.*

Definition 2 *Let $\mathcal{P}$ and $\mathcal{K}$ be clutters on a finite set E satsifying the property:*

$\forall\ E^0 \subset E$ exactly one of the following is true:
(i) $\exists\ P \in \mathcal{P}\ \ni\ P \subset E^0$
(ii) $\exists K \in \mathcal{K} \ni K \subset (E - E^0)$.

Then we call the triple $[E,\ P,\ K]$ a *blocking system.* Note that in this case the triple $[E, \mathcal{K}, \mathcal{P}]$ is a blocking system. Each of $\mathcal{P}$ and $\mathcal{K}$ are said to be *the blocker* of the other. The word "*the*" in the above will be shown to be valid later.

Lemma 1 *If $[E, \mathcal{P}, \mathcal{K}]$ is a blocking system then $P \cap K \neq \phi \; \forall \; P \in \mathcal{P}$ and $\forall \; K \in \mathcal{K}$. Conversely, if $\mathcal{P}$ and $\mathcal{K}$ are clutters on E satisfying this relation then both (i) and (ii) above can not be true.*

Proof: Let $[E, \mathcal{P}, \mathcal{K}]$ be a blocking system. If $\exists \; P$ and $K \ni P \cap K = \phi$, then taking $E^0 = P$ violates the exactly one statement. The converse is clear.

Theorem 2 *Let $\mathcal{P}$ be a clutter on E. $\exists$ a unique clutter $\mathcal{K}$ on $E \ni [E, \mathcal{P}, \mathcal{K}]$ is a blocking system. It is this that justifies the word "the" in the above.*

Proof: First we construct a clutter $\mathcal{K}$ that renders $[E, \mathcal{P}, \mathcal{K}]$ a blocking system. For this let $F = [S \subset E : S \cap P \neq \phi \; \forall \; P \in \mathcal{P}]$. Note that $F \neq \phi$. Let $\mathcal{K} = [K \in F : K' \subset K \Longrightarrow K' \notin F]$. Hence, $P \cap K \neq \phi$ $\forall \; P \in \mathcal{P}$ and $\forall \; K \in \mathcal{K}$. Thus both (i) and (ii) can not be true for this pair. Clearly $\mathcal{K}$ is a clutter on E. Suppose for $E^0 \subset E$, (i) is not true. Then $\forall \; P \in \mathcal{P}$, $(E - E^0) \cap P \neq \phi$ and hence $(E - E^0) \in F$ and hence $\exists \; K \in \mathcal{K} \ni K \subset (E - E^0)$. Hence $[E, \mathcal{P}, \mathcal{K}]$ is a blocking system.

Now we show that if $\exists \; \mathcal{K}' \; \ni [E, \mathcal{P}, \mathcal{K}']]$ is also a blocking system then $\mathcal{K} \equiv \mathcal{K}'$. Suppose not; $\exists \; K \in \mathcal{K} - \mathcal{K}'$. Consider the partition $(E - K, K)$. (ii) is satisfied with respect to $[E, \mathcal{P}, \mathcal{K}]$ and hence $\exists$ no $P \in \mathcal{P} \ni P \subset (E - K)$. Hence (i) is not satisfied for this partition for both systems and hence (ii) is for both. Hence $\exists \; K' \subset K \ni K' \in \mathcal{K}'$. *Recall* $K \notin \mathcal{K}$ and hence $K' \neq K$. Now consider the partition $(E - K', K')$. Since (ii) is satisfied for this partition for the system $[E, \mathcal{P}, \mathcal{K}']$ $\exists$ no $P \subset (E - K')$ and hence using the system $[E, \mathcal{P}, \mathcal{K}]$ $\exists \; K" \subset K' \ni K" \in \mathcal{K}$. This is a contradiction to the fact that $\mathcal{K}$ is a clutter.

Theorem 3 *Let $[E, \mathcal{P}, \mathcal{K}]$ be a blocking system and let $f : E \longmapsto \mathbf{R}$ be a function on E. Then:*

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) = \min_{K \in \mathcal{K}} \max_{e \in K} f(e).$$

Conversely, if $\mathcal{P}$ and $\mathcal{K}$ are clutters satisfying this relation for all $0/1$ valued f, then $[E, \mathcal{P}, \mathcal{K}]$ is a blocking system.

Proof: If $[E, \mathcal{P}, \mathcal{K}]$ is a blocking system, since $P \cap K \neq \phi \; \forall \; P \in \mathcal{P}, \forall K \in \mathcal{K}$, it is easy to verify that:

$$\min_{e \in P} f(e) \leq f(\hat{e}) \leq \max f(e) \; \forall \; P \in \mathcal{P}; \forall \; K \in \mathcal{K}$$

where $\hat{e} \in P \cap K$. Hence:

$$\max_{P \in \mathcal{P}} \min_{e \in P} f(e) \leq \min_{K \in \mathcal{K}} \max_{e \in K} f(e).$$

Since P and $P \in \mathcal{P}$ have finitely many elements, let $\max_{P\in\mathcal{P}} \min_{e\in P} f(e) = \min_{e\in P^*} f(e) = f(e^*)$.

Define $E^0 = [e \in E : f(e) > f(e^*)]$ and $\hat{E}^0 = [e \in E : f(e) \geq f(e^*)]$. Because of the above equation, $\exists\ P \in \mathcal{P} \ni P \subset \hat{E}^0$ (say P^*) and hence no $K \subset (E - \hat{E}^0)$. Also by the above $\exists$ no $P \subset E^0$ and hence $\exists\ K^* \subset (E - E^0)$. The equation in the theorem holds for this K^* and P^*. To show the converse, let $E^0 \subset E$. Define f as follows:

$$f(e) = \begin{cases} 1 & e \in E^0 \\ 0 & e \notin E^0 \end{cases}$$

If $\exists\ P \subset E^0$, then:

$$\max_{P\in\mathcal{P}} \min_{e\in P} f(e) = 1 = \min_{K\in\mathcal{K}} \max_{e\in K} f(e)$$

implying $K \cap E^0 \neq \phi\ \forall\ K \in K$. Hence $\exists$ no $K \in \mathcal{K} \ni K \subset (E - E^0)$. If $\exists$ no $P \subset E^0$, then $\max_{P\in\mathcal{P}} \min_{e\in P} f(e) = 0 = \min_{K\in K} \max_{e\in K} f(e)$. This in turn implies that $\exists\ K \subset (E - E^0)$. Hence the converse and hence the theorem.$\square$

1.2 Max-Flow (Min-Cut) Equality

Let A be the incidence matrix of elements of E (column) and members of P. Thus,

$$a(P,e) = \begin{cases} 1 & e \in P \\ 0 & e \notin P \end{cases}$$

Consider the LP:

$$\begin{gathered} \max \textstyle\sum_{P\in\mathcal{P}} y(P) \\ y \geq 0 \\ \textstyle\sum_{P\in\mathcal{P}} a(P,e) y(P) \leq w(e)\ \forall\ e \in E. \end{gathered}$$

$w(e)$ is to be thought of as the *capacity* of the element e. *The special case where E represents the edges (arcs) of an undirected (directed) network and P represents the set of all paths (directed paths) from a node s to a node t should be used for purposes of focusing on concrete examples.* In such a case $\mathcal{K}$ represents the cuts separating s and t.

A related problem is: $\min_{K\in\mathcal{K}} \sum_{e\in K} w(e)$. For the special case alluded to above the value of both problems are equal $\forall\ w \geq 0$ (this follows from LP duality); further, if w is integral, then the LP has optimal solutions that are (componentwise) integral.

A blocking system is said to satisfy *max-flow-min-cut equality* if the two values are equal for all $w \geq 0$. Further, if the LP has integral solutions for all nonnegative integral w then the system is said to satisfy this equality *strongly.* If we interchange the roles of $\mathcal{P}$ and $\mathcal{K}$ we get: *min-path problem* (the LP) *and min-path equality.* Not all blocking systems satisfy either of these two. For example consdier: $E = \{1, 2, ..., (2k-1)\}$ and $\mathcal{P} = \{P \subset E : |P| = k\} = \mathcal{K}$. This example occurs in *multiperson game theory and is called "majority game"*. We will show that this example does not satisfy these equalities by showing that they do not satisfy an inequality called the *length-width inequality* described below.

Let $[E, \mathcal{P}, \mathcal{K}]$ be a blocking system and let $l : E \longmapsto \mathbf{R}_+$ and $w : E \longmapsto \mathbf{R}_+$ be two functions defined on E. Let

$$\Lambda = \min_{P \in \mathcal{P}} \sum_{e \in P} l(e)$$

and

$$\Omega = \min_{K \in \mathcal{K}} \sum_{e \in K} w(e)$$

Lenght-Width inequality is said to hold for a blocking system $[E, \mathcal{P}, \mathcal{K}]$ if,

$\Lambda.\Omega \leq \sum_{e \in E} l(e).w(e) \; \forall \; l, w \in \mathbf{R}_+$. *Note that this inequality is symmetric* with respect to the role of the two clutters where as the previous two were not (although the two equalities themselves were). Not all blocking systems satisfy this inequality. However, we have the following interesting theorem.

Theorem 4 *Let $[E, \mathcal{P}, \mathcal{K}]$ be a blocking system. Then either all of the following are none of them holds:*

(i)max-flow-min-cut for the ordered pair $[P, K]$
(ii)max-flow-min-cut for the ordered pair $[K, P]$
(iii)length-width inequality for the unordered pair $\{P, K\}$.

Proof: It suffices to show that (i)$\Longleftrightarrow$(iii).

$\Longrightarrow$:Let $L(P) = \sum_{e \in P} l(e)$ and y^* be an optimal solution to the LP:

$$\begin{array}{c} \max \sum_{P \in \mathcal{P}} y(P) \\ y \geq 0 \\ \sum_{P \in \mathcal{P}} a(P, e) y(P) \leq w(e) \; \forall e \in E \end{array}$$

$\Lambda.\Omega = \Lambda. \sum_{P \in \mathcal{P}} y^*(P) \leq \sum_{P \in \mathcal{P}} y^*(P).L(P)$
$= \sum_{P \in \mathcal{P}} [y^*(P)._e \sum_{e \in P} l(e)] = \sum_{P \in \mathcal{P}} [y^*(P) \sum_{e \in E} l(e) a(P, e)]$
$= \sum_{e \in E} l(e) [\sum_{P \in \mathcal{P}} y^*(P) a(P, e)] \leq \sum_{e \in E} l(e).w(e)$

$\Longleftarrow$: Consider the dual of the LP above. It can be written as:

$$\begin{array}{c} \min \sum_{e \in E} w(e) x(e) \\ x \geq 0 \\ \sum_{e \in E} a(P, e) x(e) \geq 1 \; \forall \; P \in \mathcal{P} \end{array}$$

Let its optimal solution be $l^*(e) \geq 0$. The constraints imply that $L^*(P) = \sum_{e \in E} l^*(e) \geq 1 \; \forall \; P \in \mathcal{P}$. Since Ω represents the optimal value among integral solutions to this LP and using LP duality we get:

$$\Omega \geq \sum_{P \in \mathcal{P}} y^*(P)$$

If strict inequality holds, letting:

$$\Lambda^* = \sum_{P \in \mathcal{P}} \sum_{e \in P} l^*(e) \geq 1$$

we get a violation of length-width inequality for w and l^*. Hence the theorem.□

2
Blocking and Antiblocking Polyhedra

Much of this is based on the pioneering work of **D.R.Fulkerson**. It is mainly concerned with the well known *set packing and set covering problems* stated below for a given 0/1 matrix A and an integral vector w.:

Set Packing Problem:

$$\begin{gathered} \max\ e^t y \\ A^t y \leq w \\ y \geq 0,\ integer \end{gathered}$$

Set Covering Problem:

$$\begin{gathered} \min\ e^t y \\ A^t y \geq w \\ y \geq 0,\ integer] \end{gathered}$$

For both problems, we may assume without loss that we do not have two rows $A_{j.}$ and $A_{k.} \ni A_{j.} \geq A_{k.}$. We consider the analysis of the set packing case and its continuous version first.

2.1 Blocking Polyhedra

The LP dual of the above packing problem is:

$$\begin{gathered} \min w^t x \\ Ax \geq e;\ x \geq 0 \end{gathered}$$

The set of feasible solutions to this LP is an *unbounded polyhedron* and will be denoted by $\mathcal{B}$. $\mathcal{B}$ is the vector sum of the convex hull of its extreme points $\{b^1, b^2, ..., b^r\}$ and the nonnegative orthant $\mathbf{R}^n_+$. *All this fancy statement means is that the extreme rays are the unit vectors.* That these are the extreme rays follows from the fact that $x \geq 0 \Longrightarrow Ax \geq 0$ since $A \geq 0$.

Definition 3 *The i^{th} row, $A_{i.}$ of A is said to be* inessential *if* $\exists$:

$$\lambda \geq 0; \ni A_{i.} \geq \sum_{j \neq i} \lambda_j A_j; \sum_{j \neq i} \lambda_j = 1$$

This is equivalent to the i^{th} constraint in the definition of $\mathcal{B}$ being redundant.

Definition 4 *A is* proper *if $\exists$ no inessential rows.*

Definition 5 *The* blocker $\hat{\mathcal{B}}$ *of $\mathcal{B}$ is defined by the relation:*

$$\hat{\mathcal{B}} = \{y : y \geq 0; y^t x \geq 1 \,\forall\, x \in \mathcal{B}\}$$

The following are boring but useful results.

Theorem 5 *Let a be a proper $m \ \times n$ matrix whose rows are $\{a^i; 1 \leq i \leq m\}$. Let $\mathcal{B}$, $\hat{\mathcal{B}}$, and $\{b^k; 1 \leq k \leq r\}$ be defined as above. Let B be a $r \times n$ matrix with $B_{k.} = b^k$. Let $\mathcal{A}$ be defined by the relation:*

$$\mathcal{A} = \{y : y \geq 0; By \geq e\}$$

Then:

(i) $\hat{\mathcal{B}} = \mathcal{A}$
(ii)B is proper and the extreme points of $\mathcal{A}$ are the rows of A
(iii) $\hat{A} = \mathcal{B}$ where

$$\hat{\mathcal{A}} = [x : x \geq 0; x^t y \geq 1 \,\forall\, y \in \mathcal{A}]$$

Proof: (i) $A \subset \hat{\mathcal{B}}$: Let $y \in \mathcal{A}$ and $x \in \mathcal{B}$. Since $x \in \mathcal{B}$, $\exists\ \lambda_i \geq 0$ and $d \geq 0$ satisfying the relations:

$$x = \sum_{i=1}^{r} \lambda_i B_{i.} + d; \sum \lambda_i = 1$$

Now it is a simple matter to verify that $x^t y \geq 1$ and hence $y \in \hat{\mathcal{B}}$.
$\mathcal{B} \subset \mathcal{A}$:

$$y \in \hat{\mathcal{B}} \Longrightarrow y \geq 0; y^t x \geq 1 \,\forall\, x \in \mathcal{B}$$

$$\Downarrow$$

$$y \geq 0; y^t B_{i.} \geq 1, 1 \leq i \leq r$$

$$\Downarrow$$

$$y \in \mathcal{A}$$

(ii) B is proper: Suppose:

$$B_{1.} \geq \sum_{i=2}^{r} \alpha_i B_{i.} = y; \alpha_i \geq 0; \sum_i \alpha_i = 1$$

Let $B_{1.} = y + z; z \geq 0$. If $z = 0$, then $B_{1.}$ is not an extreme point of $\mathcal{B}$. If $z \neq 0$, then $y = \frac{1}{2}[(y + \frac{1}{2}z) + (y + \frac{3}{2}z)]$ and these are points in $\mathcal{B}$ contradicting the extreme point nature of $B_{1.}$. Hence B is proper.

Now we show that the extreme points of $\mathcal{A}$ are the rows of the matrix A. Let $\mathcal{C}$ =convex hull of $\{a^1, a^2, ..., a^m\}$. We wish to show that $\mathcal{A} = \mathcal{C} + \mathbf{R}^n_+$.

$\mathcal{A} \supset \mathcal{C} + \mathbf{R}^n_+$: Let $x \in \mathcal{C} + \mathbf{R}^n_+$.

$x = \sum_{i=1}^{m} \alpha_i a^i + z; \alpha_i \geq 0; \sum \alpha_i = 1; z \geq 0$. It is easy to verify that $Bx \geq e$ and hence $x \in \mathcal{A}$.

$\mathcal{A} = \mathcal{C} + \mathbf{R}^n_+$: If not, $\exists\ b \in \mathbf{R}^n_+$ and $\lambda \in \mathbf{R} \ni bx \geq \lambda\ \forall\ x \in \mathcal{C} + \mathbf{R}^n_+$ and $by < \lambda$ for some $y \in \mathcal{A}$ by *separation theorem in convexity.* This in turn implies that $b \geq 0$ and hence $\lambda > 0$. Hence $\frac{b}{\lambda} \in \mathcal{B}$ and therefore $\frac{by}{\lambda} \geq 1$ and this is a contradiction and hence $\mathcal{A} = \mathcal{C} + \mathbf{R}^n_+$. Thus, the extreme points of $\mathcal{A}$ are among the rows of A. Now we show that each row of A is an extreme point of $\mathcal{A}$. Suppose $A_{1.}$ is not. Then $A_{1.} = \frac{1}{2}(x+y)$ where $x = \sum_{i=1}^{m} \alpha_i A_{i.} + q \neq A_{1.}$ and $y = \sum_{i=1}^{m} \beta_i A_{i.} + p \neq A_{1.}$. Hence $\alpha_1 + \beta_1 \leq 2$. Hence

$A_{1.} = \frac{1}{2}[\sum_{i=1}^{m}(\alpha_i + \beta_i)A_{i.} + (p+q)]$.

$\geq \sum_{i=2}^{m}(\frac{\gamma_i}{1-\gamma_i})A_{i.}$ where $\gamma_i = \frac{1}{2}(\alpha_i + \beta_i)$. This is a contradiction to the hypothesis that A is proper.

$\hat{\mathcal{A}} = \mathcal{B}$: By (i) $\hat{\mathcal{A}} = \{x : xy \geq 1\ \forall\ y \in \hat{\mathcal{B}}\}$. $x \in B \Longrightarrow xy \geq 1\ \forall\ y \in \hat{\mathcal{B}}$. Hence $\mathcal{B} \subseteq \hat{\mathcal{A}}$. Since by (ii), extreme points of $\mathcal{A}$ are the rows of A, $x \in \hat{\mathcal{A}} \Longrightarrow x \geq 0; Ax \geq e$. Hence $x \in \mathcal{B}$. Hence the last part of the theorem.□

Theorem 6 *Let A and B be nonnegative, proper matrices. Let $\mathcal{A}$ and $\mathcal{B}$ be defined by:*

$$\mathcal{B} = [x : x \geq 0; Ax \geq e]$$

$$\mathcal{A} = [y : y \geq 0; By \geq e]$$

Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be defined as before. Let one of (i)$\hat{\mathcal{A}} = \mathcal{B}$ or (ii)$\hat{\mathcal{B}} = \mathcal{A}$ be true. Then, the other is also true and extreme points of $\mathcal{A}$ are the rows of $\mathcal{B}$ and those of $\mathcal{B}$ are the rows of $\mathcal{A}$.

Proof: Suppose (ii) is true. Let C be the matrix of extreme points of $\mathcal{B}$ and let $\mathcal{C} = [y : y \geq 0; Cy \geq e]$. Since $\mathcal{C}$ is $\hat{\mathcal{B}}$ by theorem 1, $\mathcal{C} = \mathcal{A}$. But by theorem 1, $\hat{\mathcal{C}} = \mathcal{B}$. Hence (i) is true. Also by theorem 1, extreme points of $\mathcal{C}$ are the rows of A. Hence the extreme points of $\mathcal{A}$ are the rows of A. Since $\mathcal{C} = \mathcal{A}$, and B is proper and so is C; hence $\mathcal{B} = \mathcal{C}$.

Pairs of matrices (A, B) that satisfy the above are called a *blocking pair.*

Definition 6 Max-min Equality *holds for the ordered pair of matrices* $(A, B) \Longleftrightarrow \min_j B_{j.}w = [\max e^t y : A^t y \leq w; y \geq 0]\ \forall\ w \geq 0.$

Definition 7 Min-min Inequality *(also known as* length-width inequality*)* *holds for the unordered pair* $\{A, B\}$ *of matrices* $\Longleftrightarrow l.w \geq [\min_i A_{i.}l].[\min_j B_{j.}w]\ \forall\ l \geq 0,\ \forall\ w \geq 0.$

Theorem 7 *Max-min equality holds for the ordered pair of matrices* (A, B) *iff it is a blocking pair; hence max-min equality for the pair* (A, B) *holds iff it holds for the pair* (B, A)*.*

Theorem 8 *A pair of proper matrices* (A, B) *is a blocking pair iff (i)*$A_{i.}B_{j.} \geq 1\ \forall\ i, j$*; and (ii)min-min inequality holds for this pair.*

Proof:(3 & 4): (a) Let A and B be a blocking pair of matrices. Then:
$[\max e^t y : y \cdot 0; A^t y \leq w] = [\min w^t x : x \geq 0; Ax \geq e]$
$= \min_{x \in \mathcal{B}} w^t x = \min_j B_{j.}.w$
Thus, max-min equality holds for this pair.
(b)Let:

$$\lambda = \min_i A_{i.}l = \min_{y \in \mathcal{A}} y^t l$$

and

$$\Omega = \min_j B_{j.}w = \min_{x \in \mathcal{B}} x^t w$$

If either of these is zero, length width inequality follows trivially. If neither is zero, then we have:

$y(l/\lambda) \geq 1\ \forall\ y \in A$ and $x(w/\Omega) \geq 1\ \forall\ x \in B$. Hence, $l/\lambda \in \hat{\mathcal{A}}$ and $x/\Omega \in \hat{\mathcal{B}} = \mathcal{A}$. Hence $(l/\lambda)(w/\Omega) \geq 1$ and hecne length-width inequality holds.

(c)Let A and B be a pair of proper matrices that satisfy the length width inequality and condition: $A_{i.}B_{j.} \geq 1\ \forall\ i, j$. Let $\mathcal{B}, \mathcal{A}, \hat{\mathcal{B}}$ and $\hat{\mathcal{A}}$ be defined as before. Then by theorem 1,
$\hat{\mathcal{B}}$ =convex hull of $\{A_1, A_2, ..., A_m\} + \mathbf{R}^n_+$ and
$\hat{\mathcal{A}}$ =convex hull of $\{B_1, B_2, ..., B_r\} + \mathbf{R}^n_+$

If $x \in \hat{\mathcal{A}}$ and $y \in \hat{\mathcal{B}}$, by the hypothesis $x^t y \geq 1$. Thus, $\hat{\mathcal{A}} \subseteq \mathcal{B}$. We wish to show that $\mathcal{B} \subset \hat{\mathcal{A}}$. This is the same as showing $b \in \mathcal{B}$, $y \in \mathcal{A} \Longrightarrow b^t y \geq 1$. Apply length width inequality to b and y :

$$b^t y \geq [\min_j B_{j.}y][\min_{i.} A_{i.}b] \geq 1$$

Hence theorem 4.

(c) Let A and B be proper matrices satisfying max-min equality. We wish to show they are a blocking pair. We use the above proof. Let $\mathcal{B}$ be defined as before. If $B_{k.} \notin \mathcal{B}$, $\exists\ w, \alpha$ satisfying the relations:

$$B_{k.}w < \alpha \leq w^t x\ \forall\ x \in \mathcal{B}$$

by separating plane theorem. By the nature of $\mathcal{B}$ (unit vectors are extreme rays) $w \geq 0$ and hence $\alpha > 0$. But duality and max-min equality imply:
$\min_{x\in\mathcal{B}} w^t x = [\max\ e^t y : y \geq 0; A^t y \leq w] = \min_j B_{j.} w$
and this is a contradiction to the above. Hence $B_{j.} \in \mathcal{B}$ for $1 \leq j \leq r$. Thus, $A_{i.} B_{j.} \geq 1\ \forall\ i, j$. Now we will show that min-min inequality holds and hence the theorem.

Let $\lambda = \min_i A_{i.} l$, and $\Omega = \min_j B_{j.} w$. By max-min equality, $\exists$

$$y^0 \ni A^t y^0 \leq w; y^0 \geq 0; e^t y^0 = \Omega$$

Hence:

$$\lambda.\Omega = \lambda(e^t y^0) = \lambda \sum y_i^0 \leq \sum (A_{i.} l) y_i^0 = (\sum A_{i.} y_i^0) l \leq w^t l$$

Hence the min-min inequality and the theorem.

Discussion:

The most interesting case is when the matrices are 0/1. *The assumption that they be proper makes the rows correspond to the incidence vectors of the members of a clutter. If A is the incidence matrix of a clutter, then $\bar{B}$, the incidence matrix of the blocking clutter is part of the blocking matrix B. In general, B may have additional rows. There are, however, examples in which $B = \bar{B}$.* **A.Lehman** *has shown that these are precisely those for which one of min-min inequality or max-min equality holds (in which case all of them hold).*

Definition 8 *Let A be the incidence matrix of a clutter. We say that* max-min equality holds strongly *for the ordered pair (A, B) if the max-min equality holds for this pair and the LP:*

$$\max e^t y$$
$$A^t y \leq w; y \geq 0$$

has integral optimal solutions for all nonnegative integral w.

Theorem 9 *A necessary (but not sufficent) condition for max-min equality to hold strongly for the pair (A, B) described above is that B be the blocker of A and B be* 0/1.

Proof:If B is not 0/1, there is a nonintegral extreme point x of $\mathcal{B}$. Let x_1 be fractional. Since x is an extreme point, there is a system of n equations that define x uniquely. Let these be :

$$\sum a_{ij} x_j = 1; 1 \leq i \leq r$$
$$x_j = 0; r+1 \leq j \leq n$$

Let $\epsilon = \min[\max_{r+1\leq j\leq n} x_j, \max_{1\leq i\leq r}(\sum_j a_{ij} x_j - 1)]$, where the minimum is over all other extreme points of $\mathcal{B}$. It is easy to see that $\epsilon > 0$. Let

$M = \lceil \frac{x_1}{\epsilon} \rceil + 1$. Let $w = e_1 + M[\sum_{j=r+1}^{n} e_j + \sum_{i=1}^{r} A_{i.}]$. Clearly $w \geq 0$ and integral. Also,

$$w^t x = x_1 + Mr < Mr + M\epsilon \leq w^t y$$

for any other extreme point y of $\mathcal{B}$. Hence the result.□

The following example shows that this condition is not sufficent.

Counter-example: **(T.C.Hu):**

Two Commodity Packing problem:

$$A = Q^6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and its blocker B :

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given a $0/1$ *matrix* A *its blocker may or may not be* $0/1$. *Even when it is* $0/1$, *strong max-min equality may hold for neither, or one or both of the systems under consideration. No further characterization exists for any of these cases.* This is in stark contrast with the case in antiblocking systems as we will show next.

2.2 Antiblocking Polyhedra

We now consider covering problems:

$$\begin{gathered} \min\ e^t y \\ A^t y \geq w; y \geq 0 \end{gathered}$$

The dual of this LP is:

$$\begin{gathered} \max w^t x \\ Ax \leq e; x \geq 0 \end{gathered}$$

Let the set of feasible solutions to this dual be denoted by $\mathcal{C}$. $\mathcal{C}$ is bounded iff A has a positive entry in each column and we will assume that this is the case from now on. If we let the extreme points of $\mathcal{C}$ be $B_{i.}; 1 \leq i \leq r$

then $\mathcal{C}$ is their convex hull. i^{th} row of A is *inessential* if the corrsponding constraint is redundant; i.e. if:

$$\exists\ \alpha \geq 0; \sum_{j\neq i} \alpha_j = 1 \Longrightarrow A_{i.} \leq \sum_{j\neq i} \alpha_j A_{j.}$$

The *antiblocker* $\hat{\mathcal{C}}$, of $\mathcal{C}$ is defined by the relation:

$$\hat{\mathcal{C}} = [y : y^t x \leq 1\ \forall\ x \in \mathcal{C}].$$

More boring but useful theorems:

Theorem 10 *Let A be nonnegative matrix with no zero columns and let $\mathcal{C}$ and $\hat{\mathcal{C}}$ be defined as above. Let the extreme points of $\mathcal{C}$ be $\{b^1, b^2, ..., b^r\}$. Let B be defined by $B_{i.} = b^i$. Then, $B \geq 0$, has no zero columns and if $\mathcal{D} = [y : y \geq 0; By \leq e]$ then $\mathcal{D} = \hat{\mathcal{C}}$ and $\hat{\mathcal{D}} = \mathcal{C}$.*

Proof: Clearly $B \geq 0$. Since there are no zero columns in A, the largest element μ_i in the i^{th} columnof A is positive. Hence one of the rows of B is the vector e_i/μ_i. Hence no column of B is the zero vector.

$\mathcal{D} \subset \hat{\mathcal{C}}$: Let $y \in \mathcal{D}$ and $x \in \mathcal{C}$. Since $x = \sum\ \alpha_i b^i, x^t y \leq 1$. Hence $y \in \hat{\mathcal{C}}$.

$\hat{\mathcal{C}} \subset \mathcal{D}$: Trivial.

$\hat{\mathcal{D}} = \mathcal{C}$: Clearly $\mathcal{C} \subset \hat{\mathcal{D}}$. Suppose $x \in \hat{\mathcal{D}} - \mathcal{C}$. Hence $\exists\ A_{i.} \ni A_{i.}x > 1$. But $A_{i.} \in \hat{\mathcal{C}}$ and hence $A_{i.} \in \mathcal{D}$. But $x \in \hat{\mathcal{D}}$ is a contradiction to this. Hence the result.□

The following pair is an antiblocking pair:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

A has no inessential rows; only the first row of B is inessential. If we start with B or its essential rows, then we get as its antiblocker *not only the rows of* A but also the vectors $0, e_i; 1 \leq i \leq 3$. Notice that if these are appended to the rows of A then these would be inessential rows. Also, these are projections of the rows of A. This is not a coincidence as shown by:

Theorem 11 *Let A be a nonnegative matrix and let $\mathcal{C}$ be defined as before. Let $b, b^1, ..., b^s$ be points of $\mathcal{C} \ni b \leq c = \sum_{i=1}^{s} \alpha_i b^i; \alpha > 0; \sum_i \alpha_i = 1$. Also, suppose b is an extreme point of $\mathcal{C}$. Then, b is a projection of b^i for some i.*

Proof:If $b = c$, then we are done. If $b = 0$, then also the result is trivial. Hence without loss, let: $b = (\beta_1, \beta_2, ..., \beta_k, 0, 0, ..., 0)$ with $\beta_i > 0$ for $1 \leq i \leq k$ and $c = (c_1, c_2, ..., c_n)$. Since b is an extreme point of $\mathcal{C}$, $\exists$ a $k \times k$ submatrix E of $A \ni$ the system: $Ex = e$ has a unique solution $x = (\beta_1, \beta_2, ..., \beta_k)$.

However, $Ey \geq Ex = e$ where $y = (c_1, c_2, ..., c_k)$. But $c \in C$ and hence $Ey \leq e$ and hence $Ey = e$. Hence $y = x$ and hence b is a projection of one of the b^i.□

Definition 9 *A pair of matrices A and $B \ni$the polyhedra $\mathcal{C}$ and $\mathcal{D}$ defined as above are an antiblocking pair are themselves an* antiblocking pair of matrices.

Theorem 12 *Let A be an $m \times n$ incidence matrix of a clutter of subsets $S^i; 1 \leq i \leq m$ of the set $\{12, ..., n\}$. Suppose a has no zero columns. Let B be its antiblocker and let $\mathcal{C}$ and $\mathcal{D}$ be defined as before. Then, the extreme points of the type discussed in theorem 7 are precisely the rows of A together with all incidence vectors of subsets of the sets S^i (which correspond to projections of rows of A).*

Theorem 13 *Let A, B be an antiblocking pair of matrices and let $\hat{A}, \hat{B}$ be obtained by deleting i^{th} from A and B respectively. These form an antiblocking pair.*

2.3 Relation Between Blocking and Antiblocking Polyhedra

Lemma 14 *Let E be a square nonsingular $0/1$ matrix. Suppose that the system: $Ex = e$ has a (unique) solution $x \geq 0 \ni e^t x > 1$. Let $\bar{E} = J - E$, where J is the matrix all of whose elements are 1 (of the same size as E). Then, the system $\bar{E}y = e$ also has a unique solution $y \geq 0 \ni e^t y > 1$.*

Theorem 15 *Let A be as in theorem 9 with no column consisting only of ones. Let B be the $r \times n$ blocking matrix of A and let $\rho_j = e^t B_{j.}$. Let $\bar{A} = J - A$ with J being the same size matrix all of whose entries are 1. Then, the antiblocker of $\bar{A}$ is the matrix whose rows are:*

$B_{j.}/(\rho_j - 1); 1 \leq j \leq r$ and $e_i; 1 \leq i \leq n$.

2.4 Antiblocking Polyhedra Generated by 0/1 Matrices

Definition 10 *Let A be $0/1$ and let B be its antiblocker. We say that* min-max equality holds strongly for the ordered pair (A, B) *if min-max equality holds for this pair of matrices and the LP:* $[\min e^t y : A^t y \geq w; y \geq 0]$ *has integral optimal solutions for all integral w for which it has optimal solutions.*

Lemma 16 *Let A be 0/1 and let B be its antiblocker. A necessary condition for the min-max equality to hold strongly for the ordered pair (A, B) is that B be also 0/1. (Here we only look at essential rows of B).*

Proof:If a row of B is not 0/1 then it is not integral and conversely. Let x be an extreme point of $\mathcal{C} \ni x_1$ is fractional. Let the equations *defining* x be:

$$\sum a_{ij}x_j = 1; 1 \le i \le r \tag{2.1}$$

$$x_j = 0; r+1 \le j \le n \tag{2.2}$$

Let:

$$J = [j : r+1 \le j \le n; e_j \le \sum_{i=1}^{r} A_{i.}]$$

$$\epsilon_1 = \min[\max_{1 \le i \le r}(1 - \sum_j a_{ij}x_j)]$$

where the min is over all extrem points not satisfying 2.1 and

$$\epsilon_2 = \min_{j \in J}[x_j]$$

where the min is over all extreme points not covered in the above and which do not satisfy (**). *Note that these two conditions cover all other extreme points except x.* Also both these quantities are positive. Let $m = \lceil (1 - x_1)/\epsilon_2 \rceil$; $M = \max[m, \lceil (1 - x_1)/\epsilon_1 \rceil]$ and $w = e_1 + M(\sum_{i=1}^{r} A_{i.}) - m(\sum_{j \in J} e_j) \ge 0$ and integral. Also:

$w^t x = Mr + x_1$ and $w^t y \le Mr - M\epsilon_1 + 1 \le Mr + x_1$ for all extreme points of the kind used in defining ϵ_1. $w^t y \le Mr + 1 - m\epsilon_2 \le Mr + x_1$ for all extreme points used in defining ϵ_2. Hence x is optimal and the value is fractional as desired. Hence the lemma.□

Surprisingly, this condition also turns out to be sufficient. Contrast this with that in blocking polyhedra.

Theorem 17 *Let A be 0/1 with no zero columns and let B be its antiblocker. Min-max equality holds strongly for the ordered pair (A, B) iff each essential row of B is 0/1. Hence, if the min-max equality holds for the ordered pair (A, B) then it also holds for the ordered pair (B, A).* This last result is known as the pluperfect graph theorem.

Proof:We need only prove the sufficiency. Suppose A and B are 0/1 and are an antiblocking pair. Clearly min-max equality holds – we are only trying ot show that it holds *strongly*. We will produce an integer optimal solution ot the LP: $[\min\ e^t y : y \ge 0; A^t y \ge w]$ for integral w. We do this by an algorithm. Using the fact that essential rows of B are the extreme points of $\mathcal{C}$ and hence these are 0/1 and using LP duality we have:

$$e^t y^* = \max_j w^t B_{j.} = \Omega$$

and since B is 0/1, Ω is an integer. *Our process is an induction on the value of* Ω. Clearly we may suppose that $w \geq 0$. If w is not > 0, then we have the same LP on a column submatrix for which the antiblocker is the corresponding column submatrix of B and the same arguement can be repeated. If $w = 0$, then $\Omega = 0$ and $y^* = 0$ is the required integer solution in the latter case. Hence we will assume without loss that $w > 0$ and hence $\Omega > 0$. We start with $y = 0$ and "build" y^* step by step. Let:

$$w^t B_{j.} \begin{cases} = \Omega & 1 \leq j \leq k \\ < \Omega & k+1 \leq j \leq r \end{cases}$$

The system:

$$x^t B_{j.} \begin{cases} = 1 & 1 \leq j \leq k \\ \leq 1 & k+1 \leq j \leq r \end{cases}$$

has a nonnegative solution w/Ω. Hence $\exists$ an extreme point, z, of the polyhedron:

$$\mathcal{Q} = \{x : x \geq 0\} \cap \{x : x^t B_{j.} \begin{cases} = 1 & 1 \leq j \leq k \\ \leq 1 & k+1 \leq j \leq r \end{cases} \}$$

But each such extreme point of $\mathcal{Q}$ is also an extreme point of $\mathcal{D}$ and hence z is a row of A or the projection of such a row. Let z be $A_{i.}$ or its projection. Increase y_i by one and change w by the relation:

$w^{new} = w^{old} - A^t_{i.}$. Delete all columns of A and B that correspond to nonpositive components in w^{new}. This yields $A', B', w' > 0$. The pair (A', B') is an antiblocking pair. Hence:

$$1 \geq A^t_{i.} B_{j.} \geq z^t B_{j.} = 1; 1 \leq j \leq k$$

$$\Downarrow$$

$$A^t_{i.} B_{j.} = 1; 1 \leq j \leq k$$
$$A^t_{i.} B_{j.} = 0/1; k+1 \leq j \leq r.$$

$$\Downarrow$$

$$\max_j (w')^t B'_{j.} = \max_j (w^t B_{j.} - A^t_{i.} B'_{j.}) = \Omega - 1$$

We now repeat the process and in this manner build up an nonnegative integer vector $y \ni$

$$A^t y \geq w; e^t y = \Omega$$

This completes the proof of the theorem.□

Remarks

1 *We do not know the equivalent process for blocking systems. See remarks there.*

2 If A is *totally unimodular*, it is clear that its antiblocker is 0/1 and hence min-max equality holds for both the pairs(A, B) and (B, A). *Note that B may not be totally unimodular. See example dealing with rigid circuit graphs.*

3 There are examples where the strong min-max equality is trivial for one order but not for the other; yet this theorem proves these results are equivalent.

4 Let A be the incidence matrix of a clutter. Let B be the incidence matrix of the antiblocker of A and let the strong min-max equality hold for the pair (A, B). *Then, A is the (clique-node) incidence matrix of a family of maximal cliques of a graph G* and B is the incidence matrix of the *family of (anticliques) independent sets (to nodes) or equivalently the clique node incidence matrix of the complement graph $\bar{G}$.*

Proof:We construct the graph G as follows. The nodes of G correspond to the columns of A. There is an edge (i, j) in $G \iff A_{.i}.A_{.j} \geq 1$. *Such a graph is called the intersection graph of the matrix A.* By the previous results, strong min-max equality $\iff B$ is 0/1 $\iff \mathcal{C}$ has 0/1 extreme points. *Note that all 0/1 points of $\mathcal{C}$ are necessarily extreme.* Let $\hat{A}$ be the clique-node incidence matrix of this graph G. Let $\hat{\mathcal{C}} = [x : x \geq 0; \hat{A}x \leq e]$.

Claim: $\hat{\mathcal{C}} = \mathcal{C}$.

Proof: For this we need to show that each inequality in one is implied by the other set. Consider the inequality $x^t A_{i.} \leq 1$. Suppose $T = [j : a_{i,j} = 1]$. This implies (i, j) is an edge of G if $j \in T$. Hence, T is a subset of some clique in G and the corresponding inequality in $\hat{\mathcal{C}}$ implies the above inequality $x^t A_{i.} \leq 1$. This implies that $\hat{\mathcal{C}} \subseteq \mathcal{C}$.

To show the converse, suppose $\mathcal{C} - \hat{\mathcal{C}} \neq \phi$. Then, $\exists$ an extreme point x^0, of $\mathcal{C}$, $\ni x^0 \notin \hat{\mathcal{C}}$. (*This is because both are bounded convex sets and one is contained in the other.*) This imples that $\exists\ i \ni (x^0)^t \hat{A}_{i.} \geq 2$ (since all extreme points of $\mathcal{C}$ are 0/1). This implies that x^0 has two components in some clique equal to 1. But two nodes k and l are in a clique only if $\exists\ i \ni a_{i.k} = a_{i,l} = 1$. In this case x^0 violates the constriant $(x^0)^t A_{i.} \leq 1$ and hence a contradiction to $x^0 \in \mathcal{C}$.

Hence B is an antiblocker not only of A but also of $\hat{A}$. Thus both these matrices contain all the *essential extreme points* of $\mathcal{D} = [y : y \geq 0; By \leq e]$. thus, all the essential rows of $\hat{A}$ are contained in A and hence A is the clique-node incidence matrix of G.□

2.5 Examples

I **Permutations:** Consider the system:

$$\sum_{i=1}^{n} x_{ij} = 1; 1 \leq j \leq n$$
$$\sum_{j=1}^{n} x_{ij} = 1; 1 \leq i \leq n$$
$$x_{ij} \geq 0$$

It is well known that the extreme points of the above system are the incidence vectors of $n \times n$ permutations matrices viewed as vectors in the space of appropriate dimension or their projections. If A is an $n! \times n^2$ matrix whose rows are the above incidence vectors, then the extreme points of $\mathcal{C} = [x : Ax \leq e; x \geq 0]$ are precisely the rows of the constraint matrix of the system that we started with. Since, this constraint matrix is totally unimodular, it follows that strong min-max equality holds in this case. If we let the constraint matrix in the above be B, strong min-max for (B, A) for $0/1$ w is a well known theorem due to **König**:

Theorem 18 *A: Let $G = [S, T; E]$ be an undirected bipartite graph. The maximum nunmber of arcs of G that are pariwise node disjoint (*matchings*) equals the minimum number of nodes in an (S, T) disconnecting set of nodes (*node covers*).*

Mathematical Programming Interpretation:

Let B be the node edge incidence matrix of G. Consider the polyhedron $\mathcal{C} = [x : Bx \leq e; x \geq 0]$. The antiblocker A has as its rows the incidence vectors of matchings. Since B is totally unimodular, A is $0/1$. For $0/1$ w, consider the $LP : [\min e^t y : y \geq 0; B^t y \geq w]$. Any $0/1$ solution is a node cover. Note

$[\min e^t y : y \geq 0; B^t y \geq w; y \text{ integral}]$
$\geq [\min e^t y : y \geq 0; B^t y \geq w]$
$= [\max w^t x : x \geq 0; Bx \leq e]$
$\geq [\max w^t x : x \geq 0; Bx \leq e; x \text{ integral}]$
$= \max_i A_{i.} w$.

Hence, if the first and the last term are equal, then everything in between is also equal to these. This in turn implies that this problem has integral solution for $0/1$ w (actually it satisfies strong min-max for all integral w).

Theorem 19 *B: Let G be as in theorem A. Then the minimum number of colours required in an edge colouring such that no two adjacent edges receive the same color equals the maximum degree of a node.*

Mathematical Programming Interpretation:

Consider the LP: $[\min e^t y : y \geq 0; A^t y \geq w]$. For $0/1$ w, any $0/1$ solution is a partition of the edges into matchings and hence a colouring. $\max_j B_{j.} w$

is the maximum degreee of a node in G. Thus, this is the strong min-max for the pair (A, B).

II **Chain Decomposition in a Partially Ordered Set:**

Let A be the incidence matrix of all chains (rows) in a *partially ordered set* of n elements. Then, B the antiblocker of A is the incidence matrix of all *antichains (mutually unrelated elements)* of the partially ordered set. To see this we note that 0/1 vectors of $\mathcal{C} = [x : Ax \leq e; x \geq 0]$ are precisely the incidence vectors of such antichains. Next we shall show that $\mathcal{C}$ has only integral extreme points by using theorem due to **Dilworth**. Incidentally, neither A nor B is even *balanced* (let alone t.u.).

Theorem 20 *Maximum number of mutually unrelated elements in a partially ordered set (the maximum cardinality of an antichain) equals the minimum number of chains in a chain decomposition (minimum number of chains required to cover the elements).*

This result is proved in network flows using max-flow-min-cut. It is equivalent to the strong min-max equality for the ordered pair (A, B) for 0/1 w with the above interpretation for the matrices. To show that it implies strong min-max for all integral nonnegative w we use a process called *replication* which we describe algebraically below and then show the corresponding changes in the partially ordered set.

Lemma 21 *Consider the LP* : $[\min e^t y : A^t y \geq w; y \geq 0]$ *with* $w_1 \geq 2$ *and any* 0/1 *matrix* A *as shown below:*

$$A = \begin{matrix} e^m & A^1 \\ 0 & A^2 \end{matrix} ; y = \begin{matrix} y^1 \\ \hat{y} \end{matrix} ; w = \begin{matrix} w_1 \\ \hat{w} \end{matrix}$$

where $e^m, y^1 \in \mathbf{R}^m; 0, \hat{y} \in \mathbf{R}^n; A^1$ *is* $m \times p; A^2$ *is* $n \times p; \hat{w} \in \mathbf{R}^n;$*and the LP:* $[\min \bar{e}^t \bar{y} : \bar{y} \geq 0;\ \bar{A}^t \bar{y} \geq \bar{w}]$ *with* $\bar{A}$ *as shown below:*

$$\bar{A} = \begin{matrix} e^m & 0 & A^1 \\ 0 & e^m & A^1 \\ 0 & 0 & A^2 \end{matrix} ; \bar{y} = \begin{matrix} \bar{y}^1 \\ \bar{y}^2 \\ y^+ \end{matrix} ; \bar{w} = \begin{matrix} 1 \\ w_1 - 1 \\ \hat{w} \end{matrix}$$

These two problems are equivalent.

Proof: Let $(y^1, \hat{y})$ be a feasible solution to the first problem above. $\{\bar{y}^1 = [(w_1 - 1)/w_1] y^1; \bar{y}^2 = y^1 - \bar{y}^1; y^+ = \hat{y}\}$ yields a corresponding feasible solution to the second with the same objective value. If the sarting solution is integral, then $y^1 \neq 0$ and hence it is greater than some unit vector; say e_1. In this case letting $\{\bar{y}^1 = e_1; \bar{y}^2 = y^1 - \bar{y}^1; y^+ = \hat{y}\}$ yields an integral solution to the second problem with the same value. To show the converse, let $y^1 = \bar{y}^2 + \bar{y}^1; y^+ = \hat{y}$.□

In the case of partially ordered set, this is equivalent to replicating an element (making a clone whose relationship is the same as the original element). This shows the results promised.

III **Rigid Circuit Graphs (triangulated graphs or chordal graphs):**

A graph is said to be *rigid circuit* if every circuit of length four or more has a chord. (see figure). Clearly, node induced subgraphs of a rigid circuit graph are also rigid circuit graphs. Let A be the clique node incidence matrix of a rigid circuit graph G. The antiblocker B of A is the node — independence set incidence matrix of G or equivalently the node-clique incidence matrix of the complement graph $\bar{G}$. There are rigid circuit graphs for which A is not *t.u.*; for example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

There are two types of *replications of nodes* in a graph. In the first type, we duplicate the vertex (meaning the clone is connected to all vertices that the original is connected to) but do not join the original vertex to its clone. In the second type we connect these two. The second type of replication preserves rigid circuit property, but the first does not as in K_3. However, such a replication of the complement of a rigid circuit graph produces a complement of a rigid circuit graph. Thus, if the strong min-max equality holds for the pair (B, A) for $0/1$ right hand sides then it also holds for all integral right hand sides. This is Berge's proof for strong min-max equality for this pair of matrices for both orders. Fulkerson gives another proof that sheds some light on the structure of A as well. It uses a result of **Dirac**.

Theorem 22 *If G is a rigid circuit graph, then $\exists$ a vertex which is* simplicial.

Definition 11 *A vertex is* simplicial *if it together with its neighbours forms a clique.*

Actually a stronger result is proved through induction. But we need a definition and a lemma first.

Definition 12 *An* articualtion set *is a set of vertices in a graph whose removal yields a disconnected graph. A* minimal *articulation set is one that is an articulation set which is minimal in the set theoretic sense.*

Lemma 23 *Let G be a rigid circuit graph and let S be a minimal articulation set. Then, the induced subgraph G_S is* complete.

Proof: Let the components when S is removed be $\Gamma_1, \Gamma_2, ...$ etc. Every node in S is connected to at least one node in each of Γ_i; else S would not be minimal. Let $s \in S$, $t \in S$. $\exists$ a chain $[s, k_1, k_2, ..., t]$ with $k_i \in \Gamma_1$. Let this be the shortest such chain. Similarly let $[s, m_1, m_2, ..., t]$ with $m_j \in \Gamma_2$ be the shortest such chain. Now consider the cycle formed by their union.

The following chords are not present: (s, k_i) : contradiction to the shortness of the first chain; (k_i, k_l) : same reason as above; (t, k_i) : same reason as above; (s, m_j) : contradiction to the shortness of the second chain; (m_j, m_r) : same reason as above; (t, m_j) : same reason as above; (k_i, m_j) : in two different components.

This circuit has a length at least four and hence must have a chord. We have ruled out all possibilities except (s, t) and hence s and t must be connected by an edge. Hence the lemma.□

Theorem 24 *Let G be a rigid circuit graph. There are two simplicial vertices (assuming the number of vertices ≥ 2); if G is not complete, then these are nonadjacent.*

Proof: By induction on the number of vertices. If G is complete there is nothing to prove. If s and t are nonadjacent vertices in G, let S be the minimal separator of s and t (removal of S yields two components G_A and G_B with $s \in G_A$ and $t \in G_B$). The lemma shows that the induced subgraph on S is complete. The graph if $G_{A \cup S}$ is complete then every vertex in A is simplicial; if it is not, then there are two nonadjacent vertices in it that are simplicial by induction hypothesis. Since G_S is complete, one of these must be in A. Similar arguement shows that there is a simplicial vertex in B and this proves the theorem.

P.Buneman and Gavril go further to prove a stronger result. This has very useful applications in location theory and file organization.

Example:

Let T be a tree and $\mathcal{T}$ be a family of subtrees of T. Let $G_{\mathcal{T}}$ be a graph fromed as follows: There is a node for each member of $\mathcal{T}$ and two of thse are connected by an edge if they intersect. It is well known in this case that, if a subfamily interscts pairwise, then they all intersect in a common point; such a property is known in the literature as the *Helly property* which also holds for a family convex sets $\in R^d$ with any $d+1$ of them intersecting (instead of two). Using this it is very easy to show that G is a rigid circuit graph. **Buneman** and **Gavril** showed the converse: every rigid circuit graph is obtained in this manner and they showed how to produce the family of trees.

Theorem 25 *(Walter (72);Buneman (74); Gavril (74): Let $G = [N; E]$ be an undirected graph. The following statements are equivalent: (i) G is a rigid circuit graph;(ii) G is the intersection graph of a family of subtrees of a tree; (iii)$\exists$ a tree $T = [K, E]$ whose vertes set K corresponds to the* *maximal cliques of* $G \ni$ *each of its induced subgraphs* T_{K_i} *is connected (and hence a subtree), where* K_i *consists of those nodes of* T *that correspond to maximal cliques in* G *containing the node* i *(in* G*).*

Proof: (iii) $\Longrightarrow$(ii): Let $(s,t) \in E \Longrightarrow \{s,t\} \subset A$ where A is some maximal clique in G. Hence $K_s \cap K_t \neq \phi$. Hence $T_{K_s} \cap T_{K_t} \neq \phi$. Thus, G is the intersection graph of the family, $\{T_{K_i} : i \in N\}$, of subtrees of T.

(ii) $\Longrightarrow$(i):

(i) $\Longrightarrow$(iii): By induction on the number of nodes in G. The result is trivial if G is complete; T is a single point. If G has several components $\Gamma_1, \Gamma_2, ..., \Gamma_p$ by induction $\exists$ trees T_i satisfying (iii) for each Γ_i. Connecting these trees toegther in any manner yields the required tree T. Hence we assume G is connected and is not complete. Choose a simplicial vertex a and let $A = \{a\} \cup$ neighbours of a. Clearly A is a maximal clique. Let $U = [i \in A : (i,j) \in E \Longrightarrow j \in A]; Y = A - U$. Since G is connected and not complete none of U, Y, $N - A$ are empty. By induction, $\exists$ a tree T_{N-U} whose vertex set K^{N-U} corresponds to maximal cliques in G_{N-U} such that for each vertex in $i \in N - U$, the set K_i^{N-U} induces a connected graph (and hence a subtree) of T_{N-U}. Note that (the set of cliques in G) $K = K^{N-U} \cup \{A\} - \{Y\}$ or $K = K^{N-U} \cup \{A\}$ depending on whether or not $\{Y\}$ is a clique in the graph G_{N-U}. Let B be a maximal clique in G_{N-U} containing Y.

Case (a): $B = Y$. We obtain tree T from T_{N-U} by letting the vertex that corresponded to B to now correspond to A.

Case (b): $B \neq Y$. We obtain tree T from the tree T_{N-U} by adding a new node (which now corresponds to A) and connecting the new node to the node representing B.

In either case $K_i = \{A\}\ \forall\ i \in U$ and $K_j = K_j^{N-U}\ \forall\ j \in N - A$, each of which induces a subtree of T. We need only check the sets K_r for $r \in Y$. In case (a), $K_r = K_r^{N-U} \cup \{A\} - \{B\}$ which induces the same subtree as K_r^{N-U} since only names were changed. In case $(b) K_r = K_r^{N-U} \cup \{A\}$, which clearly induces a subtree of T. Hence the theorem.$\square$

3
Perfect Graphs

We begin with an example due to **C.E.Shannon** who is the originator of this idea. Consider a transmitter that can send five signals: a, b, c, d, and e. Signal a might be received as p or q and so on as shown in the graph R :

This yields the "*confusion graph*" G, with nodes a through e where two nodes are connected if they can be confused at the receiving end. If G is the cycle on five nodes, we can select at most two of the five original signals if no two are to be confusing. In effect, we can choose a set only if it is *independent* in G. Instead of single letters if we choose pairs, then we have a graph with 25 nodes and it is represented by G^2 whose vertices are (i, j) with i and j from the original set of five letters. Two nodes in this graph are connected if they are of the form $[(i, j); (i, k)]$ or $[(j, i); (k, i)]$ where j and k are connected in G or if they are of the form $[(i, j); (k, l)]$ where i and k are connected in G and j and l are connected in G. G^k is defined in a similar manner. The maximum size of an independent set in G is denoted by $\alpha(G)$. It is easy to show that $\alpha(G^k) \geq [\alpha(G)]^k$. *(Shannon) Capacity*, $c(G)$, *of a graph* G is defined by the relation: $c(G) = \sup_k \alpha(G^k)^{\frac{1}{k}} = \lim_{k\to\infty} \alpha(G^k)^{\frac{1}{k}}$. This result follows from a result known in the literature as **Fekete**'s theorem which asserts: If $a_{m+n} \leq a_m + a_n$ then $\frac{a_n}{n} \to \inf(\frac{a_n}{n})$. To prove this result (See **Polya and Szego** 's book on analysis) let $\alpha = \inf(\frac{a_n}{n})$ and for $\epsilon > 0$, let $\frac{a_m}{m} \leq \alpha + \epsilon$. For large n, we have $n = qm + r$ with $0 \leq r < m$. $a_n \leq qa_m + a_r$. Hence $\frac{a_n}{n} \leq \frac{a_m}{m+\frac{r}{q}} + \beta/n$ where $\beta = \max_{1\leq j\leq m} a_j$. Hence the result follows. Using $a_n = -\log(\alpha(G^k))$ yields the result on $c(G)$.

The problem of determining Shannon capacity of even simple graphs remains unsolved. For instance, **Lovasz** solved the problem for the cycle on five nodes in 1979.We now need some definitions before we go on. Let $G = [N; E]$ be a simple undirected graph.

Definition 13 $\alpha(G)$ = *size of the largest* independent set (or anticlique) *in G; this is called the* (internal) stability number.

Definition 14 $\theta(G)$ = *minimum number of* cliques *that cover all the nodes of G. This is sometimes called the* clique covering number.

Definition 15 $\omega(G)$ = *size of the* largest clique *in G; sometimes called the* clique number *of the graph.*

Definition 16 $\gamma(G)$ = *minimum* number of independent sets required to cover all nodes *of G; this is often called the* chromatic number.

It is easy to show that $\alpha(G) \leq \theta(G)$ and $\omega(G) \leq \gamma(G)\ \forall\ G$. A graph is said to be *α-perfect* if $\forall\ S \subset N$, $\alpha(G_S) = \theta(G_S)$ where these are *node-induced subgraphs*. It is said to be *γ-perfect* if $\forall S \subset N$, $\omega(G_S) = \gamma(G_S)$. It is easy to show that γ-perfectness of $G \Longleftrightarrow$ α-perfectness of $\bar{G}$ where $\bar{G}$ is the *complement graph* of G. G is said to be *perfect* if it is both γ-perfect and α-perfect; equivalently both G and $\bar{G}$ are perfect. The (*weak perfect graph conjecture* which is now the) *perfect graph theorem* asserts that one on these implies the other. The only known *minimal imperfect graphs* are odd cycles and their complements; the *strong perfect graph conjecture* asserts that these are the only minimal imperfect graphs.

3.1 Mathematical Programming Interpretation

Let A be the clique node incidence matrix and B be the clique node incidence matrix of $\bar{G}$ (and hence the incidence matrix of independent sets or *anticliques* of G). Then:

$$\gamma(G) = [\min e^t y : y \geq 0; B^t y \geq e; y\ 0/1] = \min_{y \in \mathcal{D}_I} e^t y$$

$$\alpha(G) = \max_j B_{j.} e$$
$$\omega(G) = \max_i A_{i.} e$$

$$\theta(G) = [\min e^t y : y \geq 0; A^t y \geq e; y\ 0/1] = \min_{y \in \mathcal{C}_I} e^t y$$

where $\mathcal{C}_I$ and $\mathcal{D}_I$ are the integral vectors in these convex bodies.

Lemma 26 *γ-perfectness is equivalent to the* strong min-max equality for 0/1 w for the pair(B,A)*; α-perfectness is a similar result for the pair* (B, A).

Proof: We prove the second; the first is similar. Let G be α-perfect and let G_S correspond the induced subgraph of the given 0/1 w. Then:
$\theta(G_S) = [\min e^t y : y \geq 0; A^t y \geq w; y\ 0/1]$
$\geq [\min e^t y : y \geq 0; A^t y \geq w]$
$= [\max\ w^t x : x \geq 0; Ax \leq e]$
$\geq \max_j B_{j.} w = \alpha(G_S)$.
α-perfectness of G implies all these quantities are equal for 0/1 w. Conversely, if strong min-max holds for 0/1 w for the pair(A, B) then:
$\theta(G_S) = [\min e^t y : y \geq 0; A^t y \geq w; y\ 0/1] = \max_j B_{j.} w = \alpha(G_S)$.
Hence the lemma.□

3.2 Normal Product, Cartesian Sum and Product of Graphs:

Let $G = [N; E]$ and $H = [V; A]$ be two undirected graphs. Their *cartesian sum* denoted by $G \oplus H$ is a graph whose node set is the cartesian product, $N \times V$, of the node sets of G and H; two nodes (i, j) and (k, l) of this graph are connected iff $i = k$ and $(j, l) \in A$ or if $(i, k) \in E$ and $j = l$. Their *cartesian product* denoted by $G \times H$ also has the same node set as $G \oplus H$; (i, j) and (k, l) are connected iff $(i, k) \in E$ and $(j, l) \in A$. Their *normal product* denoted by $G.H$ has the same node set as the previous two; the set of edges here is the union of the edges of the two graphs $G \oplus H$ and $G \times H$. Note that all these operations are *commutative*. Similar definitions hold for combining more than two graphs. The following results are easily proved:

I $\omega(G \oplus H) = \max[\omega(G), \omega(H)]$

II $\gamma(G \oplus H) = \max[\gamma(G), \gamma(H)]$

III $\alpha(G \oplus H) \geq \alpha(G)\alpha(H)$

IV $\alpha(G \oplus H) \leq \min[|V|\,\alpha(G), |N|\,\alpha(H)]$

V $\theta(G \oplus H) \leq \min[|V|\,\theta(G), |N|\,\theta(H)]$

VI $\gamma(G\ x\ H) \leq \min[\gamma(G), G(H)]$

VII $\gamma(G.H) \geq \max[\gamma(G), \gamma(H)]$

VIII $\omega(G.H) = \omega(G)\omega(H)$

IX $\alpha(G.H) \geq \alpha(G)\alpha(H)$

X $\theta(G.H) \leq \theta(G)\theta(H)$

Hence $[\alpha(G)]^k \leq \alpha(G^k) \leq \theta(G^k) \leq [\theta(G)]^k$. If $\alpha(G) = \theta(G)$, then each inequality in the previous statement is an equality. Hence, α-perfect graphs $c(G) = \alpha(G) = \theta(G)$. In particular, when each signal is determined by its modulation frequency, two signals can be confused iff the corresponding intervals of frequency overlap ("linear noise"). In this case the confusion graph G is an *interval graph* which is known to be perfect. Thus, longer "*words*" do not improve the code. More on this problem later.

Strong min-max equality for all nonnegative integral w for the pair (B, A) is called γ-*pluperfection* and the same for the pair (A, B) is called α-*pluperfection*. The theorem on antiblockers is the same as: γ-*pluperfection* $\Longleftrightarrow$$\alpha$-*pluperfection*. Actually Fulkerson's result applies to any 0/1 matrix (although such matrices are related to graphs). It is easy to show that:

$$\gamma - pluperfection \implies \gamma - perfection \tag{3.1}$$

$$\alpha - pluperfection \implies \alpha - perfection \tag{3.2}$$

$$pluperfection \implies perfection \tag{3.3}$$

where the last thing on either side implies both of the previous ones. We will show:

$$\alpha - perfection \implies \alpha - pluperfection \tag{3.4}$$

which is sufficient to show that the statements on the right are equivalent. This is done using the replication lemma due to **L.Lovasz.** There is another implication of the perfect graph theorem which we take up now. It is this that made Fulkerson doubt the validity of the result.

Theorem 27 *The following statements are equivalent:*

(i) Let A be a 0/1 matrix such that:

$$\begin{gathered}\min e^t y \\ A^t y \geq w; y \geq 0\end{gathered}$$

has an integer optimum value $\forall$ 0/1 w. Then, this LP has an integer (optimal solution) and hence an integer value for all nonnegative integral w.

(ii) γ-perfection $\implies$ γ-pluperfection.

(iii) γ-perfection $\implies$ perfection.

Proof:(i) $\implies$ (ii): Let A be anticlique node incidence matrix of G
(ii) $\implies$ (iii): trivial.
(iii) $\implies$ (i): Since (iii)$\Longleftrightarrow$(iii') where
(iii')α-perfection $\implies$perfection it suffices to show that (iii') $\implies$(i).

For this we need to show that A is the clique node incidence matrix of some graph.

Lemma 28 *Let A be as in (i) above. Then A is the clique node incidence matrix of the* intersection graph *of$A$ and this graph is α-perfect.*

Proof: Let the rows of A be the incidence vectors of subsets of N. Corresponding to any three rows S_1, S_2, and S_3 let $S = \cup_{i \neq j}(S_i \cap S_j)$; $1 \leq i, j \leq 3$. Consider the incidence vector w_S of S in the above LP. Letting $y_i = \frac{1}{2}$ for $1 \leq i \leq 3$ is a fesible solution with value $\frac{3}{2}$. Hence the optimal value must be 1 and hence $S \subset T$ where T is the incidence vector of some row of A. Now we use a result of **P.C.Gilmore** to complete the lemma.

3.3 Replication

Lemma 29 *If G is α-perfect and we* replicate *some node v in G by R_1 (we do not connect v and its* clone v') *then the resulting graph G' is α-perfect.*

Proof: Case(i): v is in some maximum cardinality independent set in G. Since the addition of one vertex can not increase any number by more than one (it certainly does not decrease any number), it is clear that $\alpha(G') = \alpha(G) + 1$. Hence $\theta(G') \geq \alpha(G') \geq \alpha(G) + 1$. But $\theta(G') \leq \theta(G) + 1$ and hence $\theta(G') = \theta(G) + 1 = \alpha(G')$. Please note that this arguement applies to any node induced subgraph containing both v and v'. Those that contain only one of these or none are subgraphs of G and hence the result holds. Hence G' is α-perfect.

Case (ii): v is in no maximum cardinality independent set in G.

Let C_i, $1 \leq i \leq k$ be the minimum clique cover of the nodes in G with $v \in C_1$. Note $k = \theta(G)$. Let H be the induced subgraph with node set $v \cup \{\cup_{i \neq 1} C_i\}$ and H' be the induced subgraph on $\cup_{i \neq 1} C_i$. Since G is α-perfect so are H and H'.

Claim: $\alpha(H') = \theta(H') = k - 1$.

Proof:It is clear that cliques C_i for $2 \leq i \leq k$ cover all nodes of $H'$and hence $\theta(H') \leq k-1$. If the number was any smaller then this together with C_1 would be a smaller cover for G. Hence the claim.□

Since v does not belong to any independent set of size k in G, it is not in any such set in H either. Hence $\alpha(H) = \alpha(H') = k - 1$ and since H is α-perfect, $\theta(H) = \alpha(H) = k - 1$. Thus $\exists$ cliques D_j for $2 \leq j \leq k$ in H (and hence in G) that cover all nodes of H. Let $v \in D_2$. Hence G' is covered by $(C_1 - v + v')$, D_2, ..., D_k. Hence $\theta(G') = k = \alpha(G')$. The last result is due to the fact that v is in no independent set of size k in G so that replicating it using R_1 does not increase α. Hence the perfect graph theorem.□

3.4 Solving the LP

Please note that while the LP has integer optimal solutions, these may not be obtained by say the simplex method. Also, G may have small number of nodes but a large number of cliques and anticliques. The second comment poses a difficulty on solving *even* the continuous version of the LP.

References:

L.Lovasz: "On the Shannon Capacity of a Graph" , IEEE Trans. on Information Theory, IT-25, #1, 1979, pp 1 — 7

A.Schrijver: A Comparison of the Delsarte and Lovasz Bounds, IEEE Trans. on Information Theory, IT-25, #4, 1979, pp 425-429

Let

$\theta^*(G) = [\min e^t y : y \geq 0; A^t y \geq e] = \min_{y \in C} e^t y$

$= [\max e^t x : x \geq 0; Ax \leq e] = \alpha^*(G)$

Since $\alpha^*(G.H) = \alpha^*(G)\alpha^*(H) = \theta^*(G)\theta^*(H) = \theta^*(G.H)$ and

$\theta^*(G) \leq \theta(G) \leq \alpha(G) \leq \alpha^*(G)\ \forall\ G$ it should be clear that $c(G) \leq \alpha^*(G)$ with equality holding for perfect graphs in all of these.

3.4.1 Orthonormal Representation of Graphs

Given two vectors $v \in \mathbf{R}^m$ and $w \in \mathbf{R}^n$ the *tensor product* $u = v\#w \in \mathbf{R}^{mn}$, is the vector with $u_{i,j} = v_i w_j$. Simple algebra shows that $(x\#y)^t(v\#w) = (x^t v)(y^t w)$. Let $G = [N, E]$ be a simple graph with $|N| = n$. An *orthonoraml representation* of G is a system of vectors v^i, $1 \leq i \leq n$, of unit length (*i.e.* $\|v^i\| = 1$) satisfying the relation that if $(i,j) \notin E$, v^i and v^j are orthogonal. *The dimension of these vectors is not specified.* Nothing is said about the case when $(i,j) \in E$. Hence any graph has an orthonormal representation; for example take all vectors to be pairwise orthogonal.

Lemma 30 *If systems $\{u^i\}; 1 \leq i \leq n$ and $\{v^j\}; 1 \leq j \leq m$ are orthonormal representations of G and H respectively, then the system $\{u^i\#v^j\}; 1 \leq i \leq n, 1 \leq j \leq m$ is an orthonormal representation of $G.H$.*

Let the *value* of an orthonormal representation $\{u^i\}; 1 \leq i \leq n$ be defined by:

$\min_{c:\|c\|=1}[\max_{1 \leq i \leq n} \frac{1}{(c^t u^i)^2}]$. The vector c yielding the minimum in the above is called the *handle* of this representation. Let $\beta(G)$ be the minimum value over all representations of G (we will later show that this is achieved). Call a representation that achieves this minimum value *an optimal representation.*

Lemma 31 $\beta(G.H) \leq \beta(G)\beta(H)$.

Proof: Let $u^i; 1 \leq i \leq n$ and $v^j; 1 \leq j \leq m$ be optimal representaions of G and H respectively with handles c and d. Then $\|c\#d\| = 1$ and hence $u^i\#v^j$ is an orthonormal representaion for $G.H$. Hence:

$\beta(G.H) \leq \max_{i,j}[\frac{1}{((c\#d)^t(u^i\#v^j))^2}]$
$= \max_{i,j}[\frac{1}{(c^tu^i))^2}][\frac{1}{(d^tv^j)^2}]$
$= \beta(G)\beta(H).\square$
Equality will be shown to hold in this later on.

Lemma 32 $\alpha(G) \leq \beta(G)$.

Proof: Let $u^i; 1 \leq i \leq n$ be an optimal orthonormal representation for G with handle c. Let $S \subset N$ be a maximum cardinality independent set in G with $s = |S| = \alpha(G)$. This implies that u^i and u^j are orthogonal for every pair i and j in S. Hence
$1 = c^2 \geq \sum_{i=1}^{s}(c^tu^i)^2 \geq \alpha(G)/\beta(G).\square$

Theorem 33 $c(G) \leq \beta(G)$.

Proof: $\alpha(G^k) \leq \beta(G^k) \leq [\beta(G)]^k$ and hence the theorem.

Theorem 34 $c(C_5) = \sqrt{5}$.

Proof: It is well known that $c(C_5) \geq \sqrt{5}$. To show the reverse inequality, we show that $\beta(C_5) \leq \sqrt{5}$. To do this we exhibit an orthonormal representation that does the job. The vectors u^i correspond to the *ribs* and c is the *handle of an umbrella with five ribs* that is opened to the point where the maximum angle between the ribs is $\frac{\pi}{2}$. All vectors are oriented away from the common point. It is easy to show that $c^tu^i = 5^{\frac{1}{4}}$ (see remark below) . Hence the result.

In the attached figure, we want to calculate x. It is easy to see that:
$x = \frac{a}{\sin 36^\circ} = \frac{\sqrt{2}\sin 18^\circ}{2\sin 18^\circ}\cos 18^\circ = \frac{1}{\sqrt{2}\cos 18^\circ}$
Now we calculate $\cos 18^\circ$. If we let $\theta = 18^\circ$ then we have:
$1 = \sin 5\theta = \sin 3\theta\ \cos 2\theta + \sin 2\theta\ \cos 3\theta$
$= 2\sin 2\theta\ \cos 2\theta\ \cos\theta + \sin\theta\ \cos^2 2\theta - \sin\theta\ \sin^2 2\theta$
$= 16z^5 - 20z^3 + 5z$ where $z = \sin\theta$.
Hence z satisfies the equation
$16z^5 - 20z^3 + 5z - 1$
$= (z-1)(4z^2 + 2z - 1)^2 = 0$. Since $z \neq 1$, z satisfies the equation
$4z^2 + 2z - 1 = 0$ and hence $z = \frac{\sqrt{5}-1}{4}$. Hence $\cos\theta = \frac{\sqrt{10+2\sqrt{5}}}{4}$.
$c^tu^i = \cos\beta$ where $x = \sin\beta$. Hence
$c^tu^i = \sqrt{(1-x^2)} = \sqrt{(2\cos^2\theta - 1)/2\cos^2\theta}$
$= \sqrt{\frac{2\frac{10+2\sqrt{5}}{8} - 1}{2\frac{10+2\sqrt{5}}{8}}} = \frac{1}{\sqrt{5}}$.

3.4.2 *Calculation of* $\beta(G)$

Theorem 35 *Let $A(G)$ be the class of symmetric matrices satisfying $a_{ij} = 1$ if $i = j$ or if $(i,j) \notin E$ for a graph $G = [N, E]$. Then:* $\beta(G) = \min_{A \in A(G)}$ [*the largest eigenvalue of* A].

Proof: (i)Let $u^i; 1 \leq i \leq n$ be an optimal orthonormal representation of G with handle c. Define$A$ by:

$$a_{ii} = 1; a_{ij} = 1 - \frac{(u^i)^t u^j}{(c^t u^i)(c^t u^j)} \forall i \neq j.$$

A satisfies the conditions of the theorem. Moreoever,

$$-a_{ij} = [c - \frac{u^i}{c^t u^i}]^t [c - \frac{u^j}{c^t u^j}] \forall i \neq j$$

and:

$$\beta(G) - a_{ii} = [c - \frac{u^i}{c^t u^i}]^2 + [\beta(G) - \frac{1}{(c^t u^i)^2}]$$

These equations imply that $\beta(G)I - A$ is *positive semidefinite,* and hence the largest eigenvalue of A is at most $\beta(G)$.

Conversely let A be a matrix satisfying the above conditions and let λ be its largest eigenvalue. Then $\lambda I - A$ is positive semidefinite and hence there is a factorization of A: i.e. $\exists$ vectors $x^i; 1 \leq i \leq n$ such that $\lambda \delta_{ij} - a_{ij} = (x^i)^t (x^j)$ where δ is the *Kronecker function.* Let c be a vector of length 1 which is orthogonal to each of the x^i (*such a vector exists because λ is an eigenvalue of the matrix A*). Let $u^i = (c + x^i)/\sqrt{\lambda}$. Then, $(u^i)^2 = [1 + (x^i)^2]$; $(u^i)^t (u^j) = [1 + (x^i)^t (x^j)]/\lambda = 0 \ \forall \ (i,j) \notin E$. Hence $u^i; 1 \leq i \leq n$ is an orthonormal representaiton for G. Moreover, $\lambda = 1/(c^t u^i) \forall i$. hence $\beta(G) \leq \lambda$. This completes the prrof of the theorem.□

The next result gives a method of computing $\beta(G)$.

Theorem 36 *Let G be as before. Let:*

$\mathcal{M} = [B : B \ n \times n \ p.s.d \ symmetric] \cap [B : b_{ij} = 0 \ \forall \ i \neq j; (i,j) \in E; Tr(B) = \sum_i b_{ii} = 1]$. Then $\beta(G) = \max_{B \in \mathcal{M}} \sum_i \sum_j b_{ij} = \max_{B \in \mathcal{M}} Tr(BJ)$. *Dropping the symmetry requirement and changing* $Tr(B) = 1$ to $Tr(B) \leq 1$ *does not affect the result.*

Proof: Let A be a matrix satisfying the conditions of the previous theorem with largest eigen value $\beta(G)$. Let $B \in \mathcal{M}$. Then:

$$Tr(BJ) = \sum_i \sum_j b_{i,j} = \sum_i \sum_j a_{i,j} b_{i,j} = Tr(AB)$$

and hence:

$$Tr([\beta(G)I - A]B) = \beta(G) - Tr(BJ)$$

Here both $[\beta(G)I - A]$ and B are positive semidefinite. Let z^i, $1 \leq i \leq n$ be a set of orthonormal eigen vectors of B with eigen values corresponding to them equal to λ_i, $1 \leq i \leq n$. Note that $\lambda_i \geq 0 \ \forall \ i$. Then:

$Tr([\beta(G)I - A]B) = \sum_{i=1}^n (z^i)^t [\beta(G)I - A] B z^i$
$= \sum_{i=1}^n \lambda_i (z^i)^t [\beta(G)I - A] z^i$