Problems with Set-up times: Single machine

These problems are related to Traveling Salesperson Problem (TSP).

Problem description

$m = 1$, n jobs ; $r_j = 0 \ \forall j$

p_j : processing time of job j ; $j = 1 \dots n$.

Non-preemptive schedules.

$s_{i,j}$: set up time required to set up the machine if job j follows immediately after completion of job i
$i = 1 \dots n$
$j = 1 \dots n$.

Want to minimize $\underset{S}{C_{max}(S)}$

$$C_{max} = \underbrace{\sum_j p_j}_{\text{independent of } S} + \underbrace{\text{Sum of set up times}}_{\text{depends on } S}$$

$\therefore$ Minimizing C_{max} $\Leftrightarrow$ Minimizing total set-up time.

$\Leftrightarrow$ Find a Hamilton-path in (G, w) where $w_{ij} = s_{i,j}$.

We can convert to min Ham-cycle whose decision version: Is there a $\circlearrowleft$ with length $\leq k$?

We Know that Decision Version of this is NP-C. So we look for <u>approximation algorithms</u>

(There are exponential algorithms for Computing exact Solutions Via Branch-and-Bound
(See K.G. Murty or Ailsa Land + Doig)

<u>Special Cases</u> :

Suppose edge weights satisfy <u>triangle inequality</u> : $w_{ij} + w_{j,k} \geqq w_{i,k}$

Graph is <u>complete</u>

(Sum of any two sides of a triangle $\geqq$ the third ; in length)

There is a poly.-time algorithm that guarantees a $\frac{3}{2}$ approximation.

i.e algsol $\leq \frac{3}{2}$ opt

If triangle inequality does not hold (i.e general problem) $\frac{3}{2}$-approximation in polytime $\Rightarrow$ P = NP

So we donot expect such an algorithm.

We now describe this process.

(One group will talk about other special cases)

TSP with Triangle Inequality in undir. graphs and $w \geq 0$

This result is normally credited to N. Christofides (1976) but was independently discovered at the same time (1975) by R. N. Rao (a former Student at CWRU) and presented in the same Conference by Prof. P. Kainen (one of member of R. N. Rao's PhD committee)

The idea:

Step 1. Find a minimum weight spanning tree T in (G, w).

Step 2: Let $S \subseteq V$ be the set of nodes whose degree in T is odd.

Step 3: In (G, w) find shortest paths among all pairs of nodes in S [you can do this by doing it for all pairs in V]

Step 4 Consider an undir. graph $H = [S, U]$ where each edge in U has weight equal to the length of SP in G between ends of edge in U.

Step5 : Solve ~~Weight~~ minimum weight perfect matching in H. [Note H is a Complete graph on S and $|S|$ = even]

Step6 : Create a graph $G^2 = [V; E^2]$
Where E^2 : Consists of all edges in T and all edges in paths represented by edges in Solution of perfect matching problem.

Step7 : G^2 is a connected Eulerian graph (i.e degree of every vertex in G^2 is even)

It is well known, that there is a way to traverse edges of an Eulerian graph in a cycle that uses each edge ~~only on~~ exactly once.

Step8 : Record the order in which vertices appear in the above traversal for the first time (except the final return to Starting Vertex)

THIS IS OUR SOLUTION.

Proof of $\frac{3}{2}$ approximation

Because (G, w) satisfies triangle inequality the length of any path from i to j is greater or equal to the weight of edge (i, j).

Hence length of HAM-CYCLE produced

$$\leq \sum_{i,j \in T} w_{i,j} + \sum_{(i,j) \in M} w_{i,j}$$

↑ Matching

One feasible solution for min. wt. sp. tree is a HAM-PATH whose wt $\leq$ OPT

(i.e OPT-SOLN − one edge)

(HAM-cycle)

$$PM \leq \frac{1}{2} OPT$$

$$\therefore \underbrace{\sum_{(i,j) \in T} w_{i,j} + \sum_{(i,j) \in PM} w_{i,j}}_{} \leq \frac{3}{2} OPT$$

$\leq$

TOUR of TREE + PM

Hence the result.