(0.5)

The algorithm that proves Theorem 1, generates linear dependences $\gamma^{(0)}, \gamma^{(1)}, \ldots \gamma^{(p)}$

i.e for each $i = 0, \cdots p$

$$\sum_{j=1}^{n} \gamma_j^i v_j = 0, \quad 0 \le \gamma_j^i \le 1 \quad j = 1 \cdots n$$

Such that

(a) $0 \le \gamma_j^i \le 1, \; i = 0, 1, \cdots p; \; j = 1 \cdots n$

(b) $\sum_{j=1}^{n} \gamma_j^i v_j = 0$ [γ^i is a linear dependence]

(c) $|B_i| \le d$, where $B_i = B(\gamma^i)$ $i = 0, \cdots p$

(d) $A_{i+1} \supsetneq A_i$ where $A_i = A(\gamma^i)$ $i = 0, \cdots p$

(e) $A_p = \{1, 2, \ldots, n\}$

(f) $|(B_{i+1} \cup A_{i+1}) - A_i| \le 2d.$

Lemma (Carathéodory).

Let $\sum_{j=1}^{n} \lambda_j v_j = 0$; $\lambda_j \underset{\neq}{\geq} 0$ $j = 1 \cdots n$

$\{v_j \in R^d$ $j = 1 \cdots n$, $0 \in R^d$, λ_j scalars

Let j^*: $\lambda_{j^*} > 0$.

Then one can find in $O(n d^3)$ steps $\alpha \in R_+^n$

Such that $\sum_{j=1}^{n} \alpha_j v_j = 0$, $\alpha_j \geq 0$ $j = 1 \cdots n$

$\alpha_{j^*} > 0$

and $|\{ j : \alpha_j > 0\}| \leq d+1$.

Moreover, $\lambda_j = 0 \Rightarrow \alpha_j = 0$ as well.

[This in Linear Programming is the process of finding a Basic Feasible Solution given an arbitrary solution].

For now skip proof of this

The main algorithm

Start with $\gamma^{0} = (0, 0, \cdots, 0) \in R^n$.

$A_0 = B_0 = \phi$.

$C_0 = \{1, 2, \cdots, n\}$.

$\therefore$ $i = 0$, Conditions (a), (b), (c) hold

By I.H, suppose for $\gamma^{(s)}$, $s \geq 0$, (a), (b), (c) hold.

Now we show how to construct $\gamma^{(s+1)}$ ~~Such that~~ given that (a), (b), (c) hold for $\gamma^{(s)}$.

Case 1 $\quad \underbrace{|\bar{A}_s| = |B_s \cup C_s|}_{\text{\# of components in } \gamma^s \text{ that are not equal to 1.}} > d$

Let $\lambda_j = 1 - \gamma_j^s \, ; \; j = 1, \dots n$

Since $0 \leq \gamma_j^s \leq 1 \Rightarrow 0 \leq \lambda_j \leq 1$ [and $\lambda_j = 0$ if $\gamma_j^{(s)} = 1$]

Since $\sum_{j=1}^{n} v_j = 0$ (assumption in Theorem)

& $\sum_{j=1}^{n} \gamma_j^s v_j = 0$ by I.H (b) on γ^s

$\therefore \sum_{j=1}^{n} \lambda_j v_j = 0$. [Hence λ is a linear dependence]

Choose j^* from among $j \in B_s$ if possible (i.e $0 < \gamma_j^s < 1$) and if not from C_s (i.e $\gamma_j^s = 0$)

(Since $|B_s \cup C_s| > d$, this is possible)

Now apply Carathéodory's Lemma to get $\alpha \in R_+^n$, such that

$$\sum_{j=1}^{n} \alpha_j v_j = 0, \quad \alpha_j \geq 0 \quad j = 1 \dots n$$

$\alpha_{j^*} > 0$ and $\quad$ [and $\alpha_j = 0$ if $\lambda_j = 0$]

$|D| = |\{j : \alpha_j > 0\}| \leq d+1$

For this set D,

$$|B_s \cup D| \le 2d$$

__Pf__: If $B_s = \phi$, result follows from

$$|D| \le d+1$$

If $B_s \ne \phi$, I.H (c) says $|B_s| \le d$,

$|D| \le d+1$ (Caratheodory's Lemma)

and $j^* \in B_s \cap D$

(Hence we should not double count it in $B_s \cup D$.

Choose the largest value of t (scalar) such that $\gamma_j^s + t\,\alpha_j \le 1 \quad j = 1 \cdots n$

This value is given by

$$\boxed{t_0 \ge 0, \Leftarrow 1-\gamma_j^s \ge 0 \quad \alpha_j > 0} \quad t_0 = \min\left[\frac{1-\gamma_j^s}{\alpha_j} : \alpha_j > 0\right] \quad \left[\text{Note: } \alpha \gneq 0\right]$$

Let β be a linear dependence with

$$\beta_j = \gamma_j^s + t_0\,\alpha_j \quad j = 1, \cdots n$$

(follows from $\sum_{j=1}^{n} \gamma_j^s x_j = 0$, $\sum_{j=1}^{n} \alpha_j x_j = 0$) and

$0 \le \beta_j \le 1$ $\longrightarrow$ See next page

$\gamma_j^s \geq 0, \alpha_j \geqq 0 \Rightarrow \beta_j \geqq 0 \quad j = 1 \cdots n$
$t_0 > 0$

And by choice of t_0, $\gamma_j^s + t\alpha_j \leqq 1 \; j = 1 \cdots n$

$\beta_j \leqq 1 \quad j = 1 \cdots n.$

$\gamma_j^s = 1 \Rightarrow \lambda_j = 0, \Rightarrow \alpha_j = 0 \Rightarrow \beta_j = \gamma_j^s = 1$

$\therefore A(\beta) \supseteq A_s$

$j^* \in B_s \cup C_s, \; \therefore \gamma_{j^*} < 1, \Rightarrow \lambda_{j^*} > 0$

And Caratheodory made sure that $\alpha_{j^*} > 0$

$\{ j : \alpha_j > 0 \} \subseteq \{ j : \lambda_j > 0 \} = \{ j : j \in B_s \cup C_s \}$

So when we compute t_0, the set over which we take min of $\dfrac{1 - \gamma_j^s}{\alpha_j}$ has $\lambda_j > 0$ and

hence $1 - \gamma_j^s > 0$.

$\therefore t_0 > 0$.

$\therefore \exists$ Some index j with $\beta_j = 1$, $\gamma_j^s < 1$

$\therefore A(\beta) \supsetneq A_s$

$$\{ j : 0 \le \beta_j < 1 \} = (B_s \cup D) - \{ j : \beta_j = 1 \}. \quad (**)$$

Two Cases in further discussion

Case 1 (a) $: |B(\beta)| \le d$

$\{j : 0 < \beta_j < 1\}$

In this case, let $\gamma^{s+1} = \beta$. By above discussion,

Since $A(\beta) \supsetneq A_s$, (d) holds for $i = s$.

i.e $A_{s+1} \supsetneq A_s$

Condition (f) : $|(B_{s+1} \cup A_{s+1}) - A_s| \le 2d$

holds because of (**) above and $A_{s+1} \supsetneq A_s$ ($\because$ at least one new $\beta_j = 1$)

[(e) has only to be checked for the end.]

So γ^{s+1} satisfies (a) (b) (c), (d), (f).

Case 1 (b) $|B(\beta)| > d$.

Use Caratheodory to reduce $B(\beta)$ so that eventually we have Case 1(a).

<u>Case 2</u>: $|\bar{A}_s| = |B_s \cup C_s| \leq d$.

If $|\bar{A}_s| = 0$, we are done $s = p$.

If not, define $\gamma_j^{s+1} = 1$ for all j.

and let $s+1 = p$.

Conditions (a) (b) (c) hold, (since $\sum_{j=1}^{n} v_j = 0$)

and (d) and (f) hold for $i = p-1 = s$

Having finished computation of $\gamma^0, \ldots \gamma^p$,

and hence ~~$A^0, \ldots A^p$~~ $A_0, \ldots A_p$ where

$A_i = A(\gamma^i)$ $i = 0 \ldots p$

the required permutation corresponds to

an order in which indices $\{1, 2, \ldots n\}$
j

<u>Enter</u> these sets for the <u>first</u> time.

i.e the order of j in which $\gamma_j^i = 1$.

If several components became 1 at the

same time — choose order among

them arbitrarily. i.e for $\underbrace{A_i - A_{i-1}}$.

Now we show that this permutation

satisfies the requirements of the Theorem.

(7)

Since $\sum_{j=1}^{n} \gamma_j^i v_j = 0$ for all $i = 0, \dots p$ (b) of Theorem

Recall: $C_i = C(\gamma^i) = \{j : \gamma_j^i = 0\}$

$$\therefore \sum_{j \in A_i \cup B_i} \gamma_j^i v_j = 0 \qquad i = 0, \dots p$$

$$\therefore \sum_{j \in A_i} v_j = - \sum_{j \in B_i} \gamma_j^i v_j$$

($\gamma_j^i = 1$ for $j \in A_i$)

Because of (c):

$$\left\| \sum_{j \in A_i} v_j \right\| = \left\| - \sum_{j \in B_i} \gamma_j^i v_j \right\|$$

$$\leq d$$

Since $\|v_j\| \leq 1$ $j = 1 \dots n$, $0 \leq \gamma_j^i < 1$ $j \in B_i$

and $|B_i| \leq d$ by (c)

Let $F = \{ j_1, j_2, \dots j_k \}$ in the permutation the was obtained by the alg.

and suppose $A_i \subseteq F \subseteq A_{i+1}$

{Since $\gamma_j^i = 1$ in that order}

$$\left\| \sum_{t=1}^{k} v_{j_t} \right\| \le \left\| \sum_{j \in A_i} v_j \right\| + \left\| \sum_{j \in F - A_i} v_j \right\|$$

$$\le d + |F - A_i| \quad \text{since } \|v_j\| \le 1,\ j = 1 \ldots n$$

(II)

$$\sum_{t=1}^{k} v_{j_t} = \sum_{j \in A_{i+1}} v_j - \sum_{j \in (A_{i+1} - F)} v_j$$

$$= -\sum_{j \in B_{i+1}} \gamma_j^{i+1} v_j - \sum_{j \in (A_{i+1} - F)} v_j$$

Hence $\left\| \sum_{t=1}^{k} v_{j_t} \right\| \le |(B_{i+1} \cup A_{i+1}) - F|$: (III)

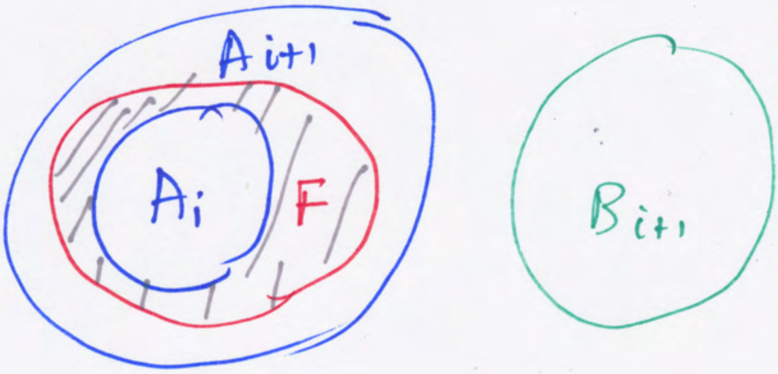
$$d + |F - A_i| + |(B_{i+1} \cup A_{i+1}) - F|$$

$$\le d + 2d = 3d$$

$$\therefore \left\| \sum_{t=1}^{k} v_{j_t} \right\| \le \frac{3}{2} d.$$

See diagram on next page for explanation

$- F - A_i$



$$\therefore \left(B_{i+1} \cup A_{i+1} - F\right) \cup \left(F - A_i\right)$$

$$= \left(B_{i+1} \cup A_{i+1}\right) - A_i$$

Time Complexity:

Condition (d) $A_{i+1} \supsetneq A_i$ implies that $p \leq n$. Hence we apply the lemma (Caratheodory) at most n times.

Complexity: $O(nd^3)$

Hence This part is $O(n^2 d^3)$.

Construction of γ^{s+1} from γ^s we must solve (basis reduction algorithms in Case 1(b) at most d times. This gives a part whose Complexity is $O(nd^4)$. Hence the total is $O(n^2 d^3 + n d^4)$.

Theorem 1 (Barany)

For a finite set $V = \{v_1, v_2, \ldots, v_n\}$ of vectors in R^d, with:

$$\sum_{j=1}^{n} v_j = 0 \quad , \quad \|v_j\| \le 1 \quad j = 1, 2, \ldots n$$

Original resut (Under any norm we use $\|x\| = \max_{i=1..d} |x_i|$.

$\exists$ a permutation $j_1, j_2, \ldots, j_n$ of $\{1, 2, \ldots n\}$ such that

$$\max_{1 \le r \le n} \left\| \sum_{k=1}^{r} v_{j_k} \right\| \le \frac{3}{2} d$$

and this permutation can be obtained in $O(n^2 d^3 + n d^4)$ Steps.

Def 1. (used in paper w/o defining)

Any solution λ to $\sum_{j=1}^{n} \lambda_j v_j = 0$ ~~called~~ is called a linear dependence.

Def 2: A linear dependence with $\lambda \neq 0$ (vector) is called a non-trivial linear dependence.

Def 3: Given a ~~...~~ linear dependence λ such that $0 \le \lambda_j \le 1$, $\sum_{j=1} \lambda_j v_j = 0$ sets $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ are defined as follows

$A(\lambda) = \{j : \lambda_j = 1\}$; $B(\lambda) = \{j : 0 < \lambda_j < 1\}$

$C(\lambda) = \{j : \lambda_j = 0\}$

~~These~~ these sets are mutualy exclusive & collectively exhanstive