Carathéodory Lemma:

Let v_j, $(j=1\dots n) \in R^d$ be given set of ~~vector~~ vectors. Let $\lambda \gneq 0$ be a nonnegative non-zero vector in R^n such that

$$\sum_{j=1}^{n} \lambda_j v_j = 0 \in R^d.$$

[Barany calls λ a non-trivial linear dependence which also happens to be non-negative]

Let $\lambda_{j^*} > 0$. Then ~~on~~ we can find in $O(nd^3)$ time a vector $\alpha \in R^n_{+}$ such that

$$\sum_{j=1}^{n} \alpha_j v_j = 0; \quad \alpha_j \geqq 0 \quad j = 1 \dots n,$$

$$\alpha_{j^*} > 0, \quad \lambda_j = 0 \Rightarrow \alpha_j = 0, \text{ and}$$

$$|\{j : \alpha_j > 0\}| \le d+1 \text{ (including } j^*\text{)}$$

Let ${}_d\overset{n}{A}$ matrix whose columns correspond to the vectors v_j; $j=1\dots n$

$$\left[\sum_{j=1}^{n} x_j v_j = 0\right] \Longleftrightarrow \left[Ax = 0\right]$$

(2)

If $x_{j^*} > 0$, $x_j \geqq 0$ $j \neq j^*$, $j = 1 \dots n$; we can rewrite $Ax = 0$ or $\sum_{j=1}^{n} x_j v_j = 0$ as follows.

$$-\sum_{j \neq j^*} x_j v_j = x_{j^*} v_{j^*}$$

and hence $$-\sum_{j \neq j^*} \frac{x_j}{x_{j^*}} \cdot v_j = v_{j^*}$$

This system looks like

$$\hat{A}\, y_j = v_{j^*} = b \qquad \rightarrow (I)$$

$$y_j \geqq 0 \quad j \neq j^*,\ j = 1 \dots n$$

Where $\hat{A}$ is $-A$ with column j^* removed.

And we are given a feasible solution y^0

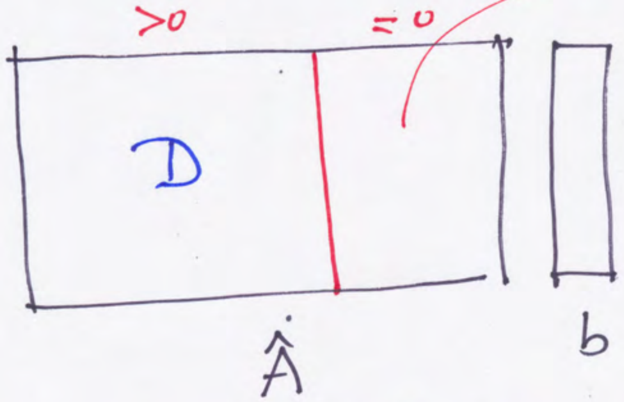
$$y_j^0 = \frac{x_j}{x_{j^*}} \qquad j \neq j^*;\ j = 1 \dots n. \quad (\cancel{\mathrm{II}})$$

Hence, $\exists$ a basic feasible solution y to System I with no more than d positive components ie $By^B = b$, $y^B \geqq 0$, where B is submatrix of A corresponding to the basis; the remaining components of $y = 0$. ($y^{NB} = 0$)

The algorithm for a finding such a Basic feasible solution maintains the property that $y_j^0 = 0 \Rightarrow y_j = 0$ as well. And all this is done in ~~$\Theta(n^3 d)$~~ $O(n d^3)$ time.

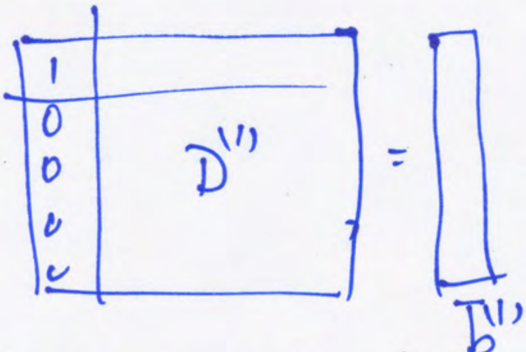
Given $\hat{A}, b, y^0$:

delete these i.e. keep these variables = 0.

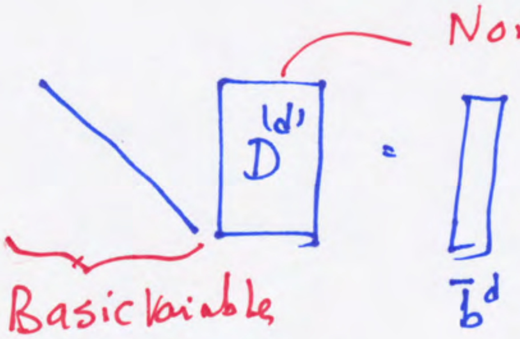


~~eg~~ we get $D(y^0)^D = b$, $(y^0)^D > 0$.

Starting "pivoting" on any non zero element of D to get a system that looks like



Keep repeating this with nonzero element in each row till we get



Canonical form
Complexity : $O(d^2 n)$

The solution is still $(y^0)^D$.

Start decreasing any <u>one</u> nonbasic variables ~~until~~ and adjusting the values of basic variables <u>until</u> either a basic variable goes down to 0 or the non ~~basic~~ variable that we decrease goes to 0. If a basic variable goes to 0, we need to pivot to exchange the role of this basic variable <u>and</u> the chosen non-basic variable that was ~~decreased~~ decreased. Whichever variable goes down to 0 is removed. The process repeated till we get a basic feasible solution.

Since $(n-d-1)$ ~~variables~~ variables have to be decreased to zero, this may involve as many pivots. Hence total complexity $O(n d^3)$

The resulting solution has all the properties promised. End of Caratheodory Lemma.