# Online Supplement to "Keyword Auctions, Unit-Price Contracts, and the Role of Commitment"

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## 1 Proof of Lemmas and Corollary

**Proof of Lemma 1.** Consider a low-performance bidder with unit valuation $v$ who bids $b$ and a high-performance bidder with unit valuation $wv$ who bids $wb$. Both bidders get a score $wb$, and their payoff functions are

$$U(y_L, v, b) = y_L(v - b)\Pr(wb \text{ is the highest score}) \tag{W1}$$

and

$$U(y_H, wv, wb) = y_H(wv - wb)\Pr(wb \text{ is the highest score}). \tag{W2}$$

It is easy to establish that

$$U(y_H, wv, wb) = \frac{wy_H}{y_L} U(y_L, v, b). \tag{W3}$$

For $b_L(v)$ and $b_H(v)$ to be equilibrium bidding functions, at any $v$, $b = b_L(v)$ must maximize $U(y_L, v, b)$ and $b = b_H(v)$ must maximize $U(y_H, v, b)$. So (W3) suggests that if bidding $b$ is the best choice for a low-performance bidder with unit valuation $v$, bidding $wb$ must be the best choice for a high-performance bidder with unit valuation $wv$, which implies $b_H(wv) = wb_L(v)$. ■

**Proof of Lemma 2.** We denote $V_H(v)$ as the equilibrium expected payoff of a bidder with performance level $y_H$ and unit valuation $v$. By (2),

$$V_H(v) \equiv U(y_H, v, b_H(v)) = y_H(v - b_H(v))\,\Pr(\text{win}|y_H, b_H(v)). \tag{W4}$$

By the Envelope Theorem, $\frac{dV_H(v)}{dv} = \frac{\partial U(y_H, v, b_H(v))}{\partial v}$. So

$$\frac{dV_H(v)}{dv} = y_H \times \Pr(\text{win}|y_H, b_H(v)) = y_H \rho_H(v), \tag{W5}$$

where the second equality is due to $\Pr(\text{win}|y_H, b_H(v)) = \rho_H(v)$ (both representing one's equilibrium winning probability). Applying the boundary condition $V_H(0) = 0$ (i.e., the bidder with the lowest valuation gets zero payoff), we get

$$V_H(v) = y_H \int_0^v \rho_H(t)dt. \tag{W6}$$

Combining (W6) and (W4) (noting $\Pr(\text{win}|y_H, b_H(v)) = \rho_H(v)$), we can solve the bidding function as

$$b_H(v) = v - \frac{\int_0^v \rho_H(t)dt}{\rho_H(v)}. \tag{W7}$$

It is easy to see that $b_H(v)$ is indeed monotonically increasing, since

$$\frac{db_H(v)}{dv} = 1 - 1 + \frac{\rho_H'(v)\int_0^v \rho_H(t)dt}{\rho_H^2(v)} > 0, \tag{W8}$$

where the inequality is due to $\rho_H'(v) > 0$.

$b_L(v)$ can be derived in the same way and is monotonically increasing. ■

**Proof of Corollary 1.** In the equilibrium, we have

$$\Delta V(v^*) = y_H \int_0^{v^*} \rho_H(t)dt - y_L \int_0^{v^*} \rho_L(t)dt = c, \tag{W9}$$

which defines $v^*$ as a function of the related parameters. Applying the Implicit Function Theory (see, e.g., Rudin, 1976, p. 223) to the above with respect to $c$ (noticing $\rho_H(t)$ is a function of $v^*$ from (21)), we have

$$\left[y_H\rho_H(v^*) - y_L\rho_L(v^*) + y_H\int_{wv^*}^{v^*}\frac{d\rho_H(t)}{dv^*}dt\right]v^{*\prime}(c) = 1. \tag{W10}$$

Since the coefficient of $v^{*\prime}(c)$ is positive, $v^{*\prime}(c) > 0$.

A similar argument leads to the conclusion that $v^*$ decreases in $y_H$. ■

**Proof of Lemma 3.** Applying the Implicit Function Theory to (W9) with respect to $w$ and noticing $\rho_L(v)$ and $\rho_H(v)$ are functions of $w$ from (20) and (21), we have

$$\left[y_H\rho_H\left(v^*\right) - y_L\rho_L\left(v^*\right) + y_H\int_{wv^*}^{v^*}\frac{d\rho_H(t)}{dv^*}dt\right]\frac{dv^*(w)}{dw} = y_L\int_0^{v^*}\frac{d\rho_L(t)}{dw}dt - y_H\int_0^{wv^*}\frac{d\rho_H(t)}{dw}dt. \tag{W11}$$

Since the right-hand side is positive (due to $\frac{d\rho_L(t)}{dw} > 0$ and $\frac{d\rho_H(t)}{dw} < 0$) and the coefficient of $\frac{dv^*(w)}{dw}$ on the left-hand side is positive, $\frac{dv^*(w)}{dw} > 0$. ■

## 2 Derivation of the Expected Revenue

The expected payment from a bidder characterized by unit valuation $v$ and performance level $y$ is the difference between his expected valuation and his expected payoff; that is

$$y_H v\rho_H(v) - V_H(v) = y_H\left[v\rho_H(v) - \int_0^v \rho_H(t)dt\right], \tag{W12}$$

$$y_L v\rho_L(v) - V_L(v) = y_L\left[v\rho_L(v) - \int_0^v \rho_L(t)dt\right]. \tag{W13}$$

The expected payment from one bidder (with probability $P_H$ having a high performance level and with probability $P_L$ having a low performance level) is

$$\begin{aligned}
& P_L E\left[y_L v\rho_L(v) - V_L(v)\right] + P_H E\left[y_H v\rho_H(v) - V_H(v)\right] \\
= \; & y_L P_L\int_0^{v^*}\left[v\rho_L(v) - \int_0^v \rho_L(t)dt\right]f_L(v)dv + y_H P_H\int_0^1\left[v\rho_H(v) - \int_0^v \rho_H(t)dt\right]f_H(v)dv \\
= \; & y_L P_L\int_0^{v^*}\rho_L(v)\left[v - \frac{1-F_L(v)}{f_L(v)}\right]f_L(v)dv + y_H P_H\int_0^1\rho_H(v)\left[v - \frac{1-F_H(v)}{f_H(v)}\right]f_H(v)dv.
\end{aligned}$$

The expected revenue from all bidders is $n$ times the above.

## 3 The Case with a Probabilistic Commitment

To specifically model bidders' concern about the auctioneer's commitment, we may assume that bidders are convinced (and it is indeed true) that with some probability the auctioneer will follow the preannounced policy and otherwise deviates to the *ex post* optimal policy. In this case, we can also show that bidders convert in a similar pattern by trading off the investment cost and the expected increase of their surplus. The auctioneer may adopt an *ex post* optimal policy in the second period, which maximizes the auctioneer's *ex post* revenue given bidders' conversion, or may follow the preannounced policy, which maximizes the auctioneer's *ex ante* expected revenue (in cases of following the commitment and of deviation in the second period).

Specifically, we assume that with probability $\delta$ the auctioneer will follow a preannounced policy. We denote $w_1^*$ as the optimal weighting factor preannounced in the first period and $w_2^*$ as the *ex post* optimal weighting factor in the second period. The cutoff $v^*$, $w_1^*$, and $w_2^*$ can be determined as follows.

First, notice $\rho_L(v)$ and $\rho_H(v)$ in (6) and (7) are functions of $w$, which can be rewritten as $\rho_L(v|w)$ and $\rho_L(v|w)$, respectively. Given a preannounced weighting factor $w_1$, the marginal bidder's break-even condition can be formulated:

$$\begin{aligned}&(1-\delta)\left[y_H\int_0^{v^*}\rho_H(t|w_2^*)dt - y_L\int_0^{v^*}\rho_L(t|w_2^*)dt\right]\\ &+\delta\left[y_H\int_0^{v^*}\rho_H(t|w_1)dt - y_L\int_0^{v^*}\rho_L(t|w_1)dt\right] = c. \qquad \text{(W14)}\end{aligned}$$

To maximize its revenue in the second period, the auctioneer chooses $w_2^*$ by letting

$$\frac{\partial\pi}{\partial w}|_{w_2^*} = 0 \qquad \text{(W15)}$$

where $\partial\pi/\partial w$ is the same as in (16). Combining the above two equations, we can derive the cutoff value and *ex post* optimal weighting factor as functions of $w_1$: $v^*(w_1)$ and $w_2^*(w_1)$.

Noting that the revenue in (14) is a function of $w$ and $v^*$, we can write the expected revenue as $E(\pi) = \delta\pi\left(w_1, v^*(w_1)\right) + (1-\delta)\pi\left(w_2^*(w_1), v^*(w_1)\right)$. By letting $\frac{d\pi}{dw}|_{w_1^*} = 0$, we can derive the optimal weighting factor preannounced in the first period.

**Example 1** *Continue with Example 3 in the paper (in which $F(v) = v$, $n = 5$, $\alpha = 0.5$, $c = 0.051$, $y_H = 1$, and $y_L$ change from 0.1 to 0.9), and let $\delta = 0.5$. Following the approach discussed above, we can derive $v^*$, $w_1^*$, and $w_2^*$, and obtain the maximum expected revenue. Figure 1 shows the revenue with the changes in the $y_L$, and compares it with those in the limited-commitment case and the full-commitment case.*

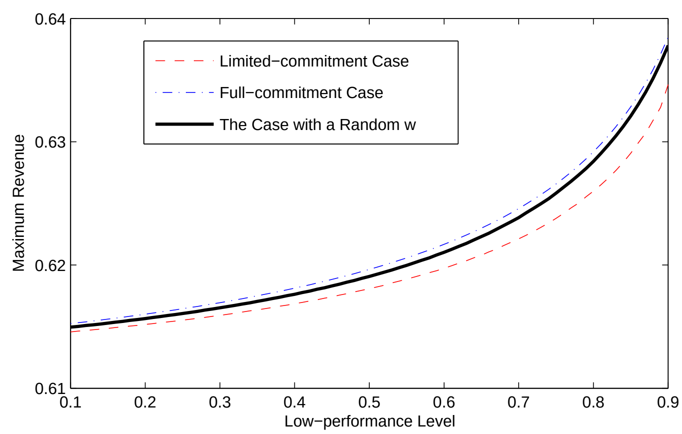


Figure 1: Revenue under the Case with a Random $w$

The example shows that the case with a random $w$ generates less revenue than the full-commitment case but more than the limited-commitment case.

# 4 Extensions

In this section, we consider relaxing some of the model assumptions.

**Correlation Between Performance and Valuation.** In our model, we assume that in the first period bidders' performance levels are independent of their unit valuations. Such an assumption might not always be realistic. One natural question is to what degree our results continue to hold if bidders' performance and valuations are correlated. For example, we may consider that in the first period the unit valuations of the low- (high-) performance bidders follow distributions $F_l(v)$ ($F_h(v)$) on $[0, 1]$. We can show that most results are robust.

First, Lemma 1 and 2 continue to hold because the proofs are distribution-free. As in Proposition 1, low-performance bidders with unit valuations above a cutoff convert to high performance in equilibrium. Now the distribution functions for low- (high-) performance bidders in the second period become

$$F_L(v) = \frac{F_l(v)}{F_l(v^*)}, \text{ if } v \leq v^*, \tag{W16}$$

and

$$F_H(v) = \begin{cases} \frac{\alpha F_h(v)}{1-(1-\alpha)F_l(v^*)} & \text{if } v \leq v^*, \\ \frac{\alpha F_h(v)+(1-\alpha)(F_l(v)-F_l(v^*))}{1-(1-\alpha)F_l(v^*)} & \text{if } v > v^*. \end{cases} \tag{W17}$$

Moreover,

$$P_L = (1-\alpha)F_l(v^*), \text{ and } P_H = 1-(1-\alpha)F_l(v^*). \tag{W18}$$

Following the same approach (as in the baseline model), it can be shown that Propositions 2 and 3 continue to hold. That is, the expected winning performance increases as the weighting factor decreases; the efficient weighting factor remains $y_L/y_H$. The intuition is the same as before. However, in the case with limited commitment, the result on the optimal weighting factor may no longer hold. This is because, as in Liu et al. (2008), when bidders' performance and valuation are correlated or bidders differ in both their performance and valuation distribution, their revenue contribution is jointly determined by both dimensions.

As a result, it becomes unclear whether the auctioneer should promote the low-performance bidders or not. However, we expect that the result in Proposition 6 continues to hold (under reasonable conditions), based on the intuition discussed in the baseline model.

**Multiple Performance Levels.** The basic intuition of our main results holds for a case with multiple performance levels. Suppose there are $k$ different performance levels, ranked from low to high and indexed by $\iota = 1, 2, ..., k$, and the weighting factor for a bidder with performance level $y_\iota$ is $w_\iota$. For simplicity, we assume that the low-performance bidders can invest to convert to the highest performance level only, and the investment cost for a bidder with performance level $y_\iota$ is $c_\iota$.

We can show, as in Proposition 1, that for each group of bidders characterized by a performance level $y_\iota$, there is an equilibrium cutoff $v_\iota^*$ such that the bidders with valuations above the cutoff convert. As in Lemma 1, a bidder with a performance level $y_{\iota_1}$ and unit valuation $v$ ties with a bidder with a performance level $y_{\iota_2}$ and unit valuation $w_{\iota_1} v / w_{\iota_2}$ in equilibrium. We can obtain $k$ equilibrium bidding functions in the same way as in Lemma 2, one for each performance level. Analogous to the case with two performance levels, the efficient weighting factors satisfy $\frac{w_1}{y_1} = \frac{w_2}{y_2} = ... = \frac{w_k}{y_k}$; that is, it is still efficient to weight bidders' bids according to their performance levels. The revenue-maximizing weighting scheme is more complex because of additional undetermined design parameters; but the basic intuition follows through, such as favoring low-performance bidders for the sake of promoting competition.

**Multi-Period Model.** We have used a stylized two-period model to derive insights. A setting with multiple periods (more than two) is more complicated. With some assumptions, however, our results continue to hold. For example, consider an $m$-period model in which the investment cost is independent of time, and once they convert, bidders' performance level will remain the same during the rest of the periods. Assume bidders' unit valuations are not revealed to each other in the earlier periods. (Otherwise, the auction becomes a dynamic one, which itself is a challenging issue in auction theory.) We can expect that low-performance bidders will convert to high performance only in the first period, if it is in their interest to convert at all. We can then observe a similar conversion pattern from the bidders characterized by low performance: Bidders with a unit valuation above some cutoff choose to convert to high performance, and the other low-performance bidders choose not to do so. In making their decision on converting, low-performance bidders now compare the investment cost and the increase of their discounted surplus flow from the conversion for the following $m-1$ periods; that is, they compare $c$ and $\frac{\beta-\beta^m}{1-\beta}[V_H(v)-V_L(v)]$, where $\beta$ is the discount rate. The efficient weighting factor is the same as in the two-period model. Similar insights also hold for the optimal design.

A multi-period model can certainly involve more complexities. For example, bidders might accumulate more knowledge as they get more involved with the auctions, so their investment cost can decrease over time. As a result, determining in which period to invest itself becomes a complicated decision that depends on the tradeoff between the opportunity for benefit and the decrease in the investment cost. Also, we can consider the case where bidders can keep their performance level only for a certain duration after they invest. In this case, as *ex post* bidders may invest at different periods, we can expect more complex dynamics related to bidders' conversion, which is an interesting extension for future research.

# References


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