

$p_x + ip_y$ Superfluid from s -Wave Interactions of Fermionic Cold Atoms

Chuanwei Zhang,^{1,2} Sumanta Tewari,^{1,3} Roman M. Lutchyn,^{1,4} and S. Das Sarma¹

¹*Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742, USA*

²*Department of Physics and Astronomy, Washington State University, Pullman, Washington 99164, USA*

³*Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29634, USA*

⁴*Joint Quantum Institute, Department of Physics, University of Maryland, College Park, Maryland 20742, USA*

(Received 30 May 2008; revised manuscript received 5 September 2008; published 17 October 2008)

Two-dimensional ($p_x + ip_y$) superfluids or superconductors offer a playground for studying intriguing physics such as quantum teleportation, non-Abelian statistics, and topological quantum computation. Creating such a superfluid in cold fermionic atom optical traps using p -wave Feshbach resonance is turning out to be challenging. Here we propose a method to create a $p_x + ip_y$ superfluid directly from an s -wave interaction making use of a topological Berry phase, which can be artificially generated. We discuss ways to detect the spontaneous Hall mass current, which acts as a diagnostic for the chiral p -wave superfluid.

DOI: [10.1103/PhysRevLett.101.160401](https://doi.org/10.1103/PhysRevLett.101.160401)

PACS numbers: 03.75.Ss, 03.65.Vf, 03.67.Lx, 73.43.Fj

Introduction.—In recent years, the physics of the 2D chiral p -wave ($p_x + ip_y$) superfluids has attracted much attention [1] because of its nontrivial statistical properties [2] and potential application in topological quantum computation [3,4]. The chiral superfluid can also act as a test bed for studying the true quantum phenomena such as quantum teleportation and violation of Bell's inequality [4,5], which are often masked by the many-body effects in a macroscopic system. There has been considerable evidence that the symmetry of the superconducting order parameter in strontium ruthenate (Sr_2RuO_4) is spin-triplet $p_x + ip_y$ [6], but the observation of exotic properties such as quantum half-vortices and non-Abelian statistics is a serious problem in Sr_2RuO_4 because of intrinsic spin-orbit coupling in the p -wave order parameter [7]. With the recent observations of p -wave Feshbach resonances in spin-polarized ^{40}K and ^6Li atoms [8–11], a $p_x + ip_y$ superfluid of fermionic cold atoms may also be realizable in the future [12–14]. Schemes for observing anyonic statistics and implementing topological quantum computation using vortices in these systems have recently been proposed [3,4]. A potential advantage of the cold atom $p_x + ip_y$ superfluid is that, being spin-polarized, non-Abelian statistics associated with the Majorana mode in the vortex cores should be readily observable.

In spite of its promise, because of the short lifetimes of the p -wave pairs and molecules in experiments [11], realizing a chiral p -wave superfluid from the p -wave Feshbach resonances [8–11] seems, at present, challenging. To circumvent this problem, in this Letter we propose to use the much more commonplace, attractive s -wave Feshbach resonances, coupled with an artificially generated topological Berry phase [15], to create a $p_x + ip_y$ superfluid of fermionic cold atoms. The topological Berry phase, in principle, can be generated in a variety of ways. Here we consider the Berry phase originating from an effective,

artificial spin-orbit coupling of atoms. In ultracold atomic gases, the effective spin-orbit coupling can be implemented by having the atoms move in spatially varying laser fields [16–20]. Since the s -wave Feshbach resonances have already been successfully used to create s -wave superfluids, our method offers a promising new way to create a topological chiral p -wave superfluid directly from the s -wave interactions. We also stress that, using the methods and ideas described in this Letter, it should be possible to realize more complex, chiral d -wave ($d_{x^2-y^2} + id_{xy}$) superfluid order parameter starting with p -wave attractive interactions.

Once a $p_x + ip_y$ superfluid is realized in experiments, one natural and important question is how to observe the chirality of the order parameter. In the superconducting state of Sr_2RuO_4 , it has been done through the observation of a nonzero polar Kerr effect [6], which demonstrates macroscopic time-reversal symmetry breaking of the $p_x + ip_y$ order parameter. However, such methods cannot be used to detect the neutral superfluid order parameter, because superfluids, in contrast to superconductors, do not directly couple to an electromagnetic field. Here we show that it is possible to detect the spontaneous Hall mass current, a clear diagnostic of the $p_x + ip_y$ order parameter, by coupling the neutral atoms to effective “electric fields” generated by optical potentials of laser fields. The method to create a $p_x + ip_y$ superfluid directly from an s -wave interaction, coupled with the methods to observe the spontaneous Hall transverse mass current, gives a complete description of a promising new way to create and analyze a chiral p -wave superfluid in fermionic optical traps.

$p_x + ip_y$ superfluid from s -wave interactions.—We consider N fermionic cold atoms confined to a quasi-two-dimensional (xy plane) trap. The atomic dynamics along the z axis are frozen out by optical traps with a high trapping frequency [21] or an optical lattice with high

potential depths [22]. The Hamiltonian of the system is

$$H = \sum_{i=1}^N \left[\frac{\mathbf{p}_i^2}{2m} - \mu + U(\mathbf{r}_i) + h_0 \sigma_i^z + H_i^{\text{so}} \right] + \frac{1}{2} \sum_{i \neq j} V_{\uparrow\downarrow}(\mathbf{r}_i - \mathbf{r}_j), \quad (1)$$

where $U(\mathbf{r}_i)$ is the external magnetic harmonic trap potential on the xy plane. We assume that $U(\mathbf{r}_i)$ is weak and the system can be taken as spatially uniform. $V_{\uparrow\downarrow}(\mathbf{r}_i - \mathbf{r}_j) = g\delta(\mathbf{r}_i - \mathbf{r}_j)$ is the attractive s -wave interaction ($g < 0$ is the interaction strength) between the atoms with opposite spins. In this BCS regime, molecules are not energetically preferred, therefore their number is strongly suppressed and their effects are negligible. μ is the chemical potential and h_0 is an effective Zeeman field for the atoms. $H_i^{\text{so}} = \gamma(\mathbf{p}_i \times \sigma_i) \cdot \mathbf{e}_z$ is the Rashba-type effective spin-orbit coupling [16], where $\sigma_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ is a vector whose components are the Pauli matrices, and γ is the spin-orbit coupling strength.

The spin-dependent part $h_0 \sigma_i^z + H_i^{\text{so}}$ in Eq. (1) can be engineered in a variety of ways. For instance, ultracold atoms moving in a 2D spin-dependent hexagonal optical lattice [16], or in a non-Abelian gauge potential created by spatially varying laser fields [19,20], will experience an effective spin-orbit coupling. The spin \uparrow and \downarrow in the effective Hamiltonian (1) denote the effective spins. Their definitions depend on the way to create the effective spin-orbit coupling and already include the spatial dependence of the lasers for generating the effective spin-orbit coupling [16,19,20]. Therefore, the corresponding laser parameters, such as the intensity and the optical lattice spacing, do not appear in the effective Hamiltonian (1) explicitly, although they do affect the parameters γ and h_0 . We note here that our proposal to generate the p -wave interactions does not depend on the specific methods to generate the effective spin-orbit coupling. Because of that, we have not specified the definitions of the effective spins in this Letter. In the experiments, h_0 , γ , and g can be adjusted by varying the laser parameters and the Feshbach resonance.

Since we consider only the short range attractive interaction, the Fourier transform of the two-body interaction is approximately a constant, $V(\mathbf{q}) = g$. Performing second quantization we obtain

$$\begin{aligned} V_{\mathbf{k}'_1 \mathbf{k}'_2; \mathbf{k}_1 \mathbf{k}_2} &= \langle \mathbf{k}'_1 \mathbf{k}'_2 | V_{\uparrow\downarrow}(\mathbf{r}_i - \mathbf{r}_j) | \mathbf{k}_1 \mathbf{k}_2 \rangle \\ &= g \int \frac{d^2 q}{(2\pi)^2} \delta_{\mathbf{k}'_1, \mathbf{k}_1 + \mathbf{q}} \delta_{\mathbf{k}'_2, \mathbf{k}_2 - \mathbf{q}} \langle u_{n\mathbf{k}'_1} | u_{n\mathbf{k}_1} \rangle \\ &\quad \times \langle u_{n\mathbf{k}'_2} | u_{n\mathbf{k}_2} \rangle, \end{aligned} \quad (2)$$

where $|u\rangle$'s are the single particle eigenfunctions of the one-body part of the Hamiltonian given by Eq. (1), and n is the band index. (In the presence of optical lattice potentials, and in the absence of spin-orbit coupling, $|u\rangle$ is simply the periodic part of the Bloch wave function.) Specializing to the BCS reduced Hamiltonian and ignoring

the residual interactions,

$$V_{\mathbf{k}, \mathbf{k}'} = g \langle u_{n\uparrow}(\mathbf{k}') | u_{n\uparrow}(\mathbf{k}) \rangle \langle u_{n\downarrow}(-\mathbf{k}') | u_{n\downarrow}(-\mathbf{k}) \rangle. \quad (3)$$

Note that, for slowly varying scattering potentials $V_{\uparrow\downarrow}(\mathbf{r}_i - \mathbf{r}_j)$ [i.e., the Fourier transformation $V(\mathbf{q})$ is nonzero only for small $q = |\mathbf{k}' - \mathbf{k}|$], the multiplicative factor on the right-hand side of Eq. (3) can be related to the sum of the Berry phases, $\Phi_{\uparrow(\downarrow)} = \int_{\mathbf{k}'}^{\mathbf{k}} \langle u_{\uparrow(\downarrow)}(\mathbf{k}'') | \frac{\partial}{\partial \mathbf{k}''} | u_{\uparrow(\downarrow)}(\mathbf{k}'') \rangle \cdot d\mathbf{k}''$, for the up and the down spins [23]. However, for the scattering potentials which are not slowly varying, $V(\mathbf{q})$ takes nonzero value even for large $|\mathbf{k}' - \mathbf{k}|$ (the situation considered in this Letter), and the extra phases renormalizing the interaction are the Pancharatnam geometric phases [24]. In the following, we show that with a Rashba type of spin-orbit coupling, one can create an effective $p_x + ip_y$ pairing interaction $V_{\mathbf{k}, \mathbf{k}'}$ from the s -wave interaction.

In the presence of Rashba spin-orbit coupling, the one-body part of the Hamiltonian can be written as

$$H_o = \begin{pmatrix} \varepsilon_{\mathbf{k}} - \mu + h_0 & -\gamma(k_y - ik_x) \\ -\gamma(k_y + ik_x) & \varepsilon_{\mathbf{k}} - \mu - h_0 \end{pmatrix}, \quad (4)$$

where $\varepsilon_{\mathbf{k}}$ is the single particle kinetic energy. The two bands of the Hamiltonian (4) corresponding to the eigenvalues $E_{\pm}(\mathbf{k}) = \varepsilon_{\mathbf{k}} - \mu \pm \sqrt{h_0^2 + \gamma^2 k^2}$ are separated by a gap, $2h_0$, at $\mathbf{k} = 0$. We choose μ at the middle of the gap such that the Fermi surface, the locus in k space where $E_-(\mathbf{k}) = \mu$, lies in the lower spin-orbit band only with the Fermi momentum denoted by k_F . We limit to the regime $\gamma k_F \gg h_0 \gg E_c(g)$, which necessitates the diagonalization of the spin-orbit coupling energy as the first step, and work with the lower band only, where $E_c(g)$ is the energy scale of the BCS pairing gap, determined by the s -wave scattering strength.

Using the appropriate eigenfunction of Eq. (4), it is straightforward to show

$$V_{\mathbf{k}, \mathbf{k}'} = g f(k, k') \exp(i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}'})), \quad (5)$$

where $f(k, k') = k'k/N_{\pm}^2(k')N_{\pm}^2(k)(\alpha + \sqrt{\alpha^2 + k^2}) \times (\alpha + \sqrt{\alpha^2 + k'^2})$, $N_{\pm}(k)$ is the normalization constant for the lower-band eigenfunction, $\theta_{\mathbf{k}}$ is the polar angle in momentum space, and $\alpha = h_0/\gamma$.

BCS pairing occurs on the Fermi surface. In the physical regime $\gamma k_F \gg h_0$ considered in this Letter, we have $k_F \gg \alpha$. In this limit, one can show

$$V_{\mathbf{k}, \mathbf{k}'} \approx g \exp(i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}'}))/4. \quad (6)$$

The interaction in the angular momentum channel m is given by $u_m(\mathbf{k}, \mathbf{k}') = \frac{1}{2\pi} \int_0^{2\pi} d\beta V_{\mathbf{k}, \mathbf{k}'} e^{im\beta}$, where $\beta = \theta_{\mathbf{k}'} - \theta_{\mathbf{k}}$ is the angle from \mathbf{k} to \mathbf{k}' . We therefore have

$$u_1(\mathbf{k}, \mathbf{k}') \approx g/4. \quad (7)$$

We see that the bare s -wave interaction at $m = 0$ channel is now completely replaced by the p -wave interaction at

$m = 1$ angular momentum channel in the physical regime $\gamma k_F \gg h_0 \gg E_c(g)$. Consequently, the pairing interaction is renormalized to a separable interaction in the p -wave channel leading to a ground state of a 2D chiral p -wave superfluid with $p_x + ip_y$ symmetry of the order parameter: $\Delta(\mathbf{p}) = \Delta_0(p_x + ip_y)/p_F$, where Δ_0 is the energy gap in the excitation spectrum of the superfluid.

The physical origin of the renormalization of the interaction may be understood through the Berry phase effects of a Rashba type of spin-orbit coupling. In the presence of a Rashba type of spin-orbit coupling, an atom evolving adiabatically in the momentum space accumulates a geometric (Berry) phase associated with the adiabatic change of the momentum \mathbf{k} , in analogy to the Aharonov-Bohm phase acquired by an electron moving in the real space in the presence of a magnetic field. Here the corresponding magnetic field in the momentum space is the Berry curvature field

$$\Omega_{\mathbf{k}} = \left[\nabla_{\mathbf{k}} \times \langle u_{\mathbf{k}} | i \frac{\partial}{\partial \mathbf{k}} | u_{\mathbf{k}} \rangle \right] \cdot \mathbf{e}_z = \frac{1}{2} \frac{\alpha}{(\alpha^2 + k^2)^{3/2}}. \quad (8)$$

The effective ‘‘magnetic flux’’ passing through the Fermi disk is $\Phi_B = \int \frac{d^2k}{(2\pi)^2} \Omega_{\mathbf{k}} \approx \pi$ as $k_F \gg \alpha$, which means a geometric phase $\beta\Phi_B/2\pi$ is obtained for the adiabatic moving of atoms from \mathbf{k} to \mathbf{k}' ($\beta = \theta_{\mathbf{k}'} - \theta_{\mathbf{k}}$). This geometric phase is the origin of the additional phase factor $\exp(-i\beta\Phi_B/\pi)$ in the interaction $V_{\mathbf{k},\mathbf{k}'}$ [Eq. (6)] around the Fermi surface and leads to the p -wave pairing at $m = 1$ channel. Remarkably, if originally the bare interaction is in the $p_x + ip_y$ channel, the Berry phase renormalizes the interaction to the d -wave channel ($m = 2$). In this way a 2D $d_{x^2-y^2} + id_{xy}$ superfluid should be realizable, which is very difficult to create using the conventional Feshbach resonance approach.

In experiments, one can choose a suitable attractive interaction regime (BCS side) so that the pairing gap for the s -wave superfluid would be $\sim \hbar \times 200$ Hz. The laser parameters for generating the effective spin-orbit coupling should be chosen so that the Zeeman field h_0 is $\sim \hbar \times 1$ kHz. For a typical Fermi energy $E_F \sim \hbar \times 1$ kHz [11], the spin-orbit coupling constant should be chosen so that $\gamma k_F \sim \hbar \times 10$ kHz, which should be achievable within the current experimental technology [16,19,20]. With these parameters, we can limit our discussion to the lower spin-orbit energy band and create a $p_x + ip_y$ superfluid from the s -wave attractive interaction using the methods described earlier.

Transverse Hall mass current in 2D $p_x + ip_y$ superfluid.—Neutral atoms in superfluid can interact with laser fields through dipole interactions [25]. The dipole interaction can provide an optical potential, whose gradient can be taken as an ‘‘effective electric field’’ for the atoms. Here, we study the linear response of a chiral $p_x + ip_y$ fermionic superfluid subject to such external effective electric fields

which act as a perturbation. The following two types of external effective electric fields will be considered.

First, we consider an effective electric field E_y , applied along the y direction, and calculate the transverse response of the superfluid along the x direction. This transverse response, which gives rise to a spontaneous Hall mass current, is a clear diagnostic of the broken time-reversal invariance and the associated chirality of the $p_x + ip_y$ order parameter. The transverse Hall current changes the sign as the chirality of the order parameter is reversed. As such, this mass current can be used to detect the realization of the chiral p -wave superfluid. In experiments, this effective electric field, $\mathbf{E} = -\nabla V(\mathbf{r})$, can be realized by applying a perturbation potential $V(\mathbf{r}) = V_0 \exp(-y^2/2\chi^2)$ created by a laser beam traveling along the x direction, where χ is the beam waist of the laser. For simplicity, we set the temperature $T = 0$ and neglect finite temperature effects. We also assume that the external trap potential is very weak, and neglect the effects of the spatial inhomogeneity.

The antisymmetric component of the spontaneous Hall conductivity, $\sigma_{xy} = -\sigma_{yx}$, for the chiral superfluid can be obtained from the anomalous chiral response coefficient, which leads to (in momentum and frequency domain) [26]

$$\sigma_{xy}(\mathbf{q}, \omega) \approx q^2/2hd(2\omega^2/v_F^2 - q^2), \quad (9)$$

in the low frequencies $\omega \ll \Delta_0$ region, where d is the thickness of the superfluid along z direction, h is the Planck constant, v_F is the Fermi velocity. Since we consider a time independent perturbation potential, Eq. (9) can be simplified to $\sigma_{xy} \approx -1/2hd$.

Equation (9) can be used to calculate the transverse mass current $j_x(\mathbf{r}, t) = \iint \sigma_{xy}(\mathbf{q}, \omega) E_y(\mathbf{q}, \omega) e^{i\mathbf{q}\cdot\mathbf{r}} e^{i\omega t} d\mathbf{q} d\omega$ induced by the longitudinal effective electric field E_y . For the potential $V_0 \exp(-y^2/2\chi^2)$, this can be simplified as

$$j_x(\mathbf{r}) = V_0 y \exp(-y^2/2\chi^2)/2hd\chi^2. \quad (10)$$

We see that at the peak of the potential, $y = 0$, the mass current is zero. The current flows in opposite directions on the two sides of the potential, and reaches maximum at $y = \pm\chi$.

In the time-of-flight measurements, such a current would lead to a velocity of BCS pairs, $v_x(\mathbf{r}) = j_x(\mathbf{r})/n(\mathbf{r})$, where $n(\mathbf{r})$ is the density of Cooper pairs. The different velocities of the atoms on the two sides yield a larger image along the x direction compared to the unperturbed case. The enhancement of the size of the image is determined by the maximum transverse velocity, $v_x^{\max} = V_0 e^{-0.5}/2n\chi h$, which occurs for the atoms at $y = \pm\chi$. Assuming representative values for the parameters, $\chi = 20 \mu\text{m}$, $V_0/h = 100$ Hz, the 2D Cooper pair density $n = 10^{12} \text{ m}^{-2}$, we find $v_x^{\max} = 8 \mu\text{m/s}$. For a time of flight 500 ms, the enlargement of the image is about $4 \mu\text{m}$, which should be observable in experiments.

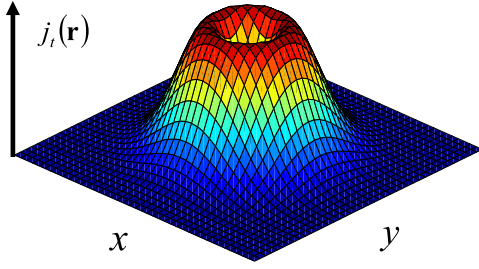


FIG. 1 (color online). Generating a rotating 2D $p_x + ip_y$ superfluid through a nonrotating laser propagating along the z direction. The figure shows the tangential current distribution $j_t(\mathbf{r})$ for a static laser potential $V(\mathbf{r}) = V_0 \exp(-r^2/2\chi^2)$. The current has the maximum at $r = \chi$ and decreases on both sides, where χ is the laser beam waist.

In the experiments, one can also detect the velocity distribution $v_x(y)$ directly by the two-photon Raman transition [25]. The Raman lasers are focused on a local region and transfer atoms in that region to another hyperfine state $|3\rangle$. Then one can detect atoms at the state $|3\rangle$ using the time-of-flight image, leading to a determination of their velocity distribution.

The second type of effective electric field we consider is generated by a laser beam propagating along the z direction and centered at $(x, y) = (0, 0)$. The optical potential can be written as $V = V_0 \exp(-r^2/2\chi^2)$, where $r = \sqrt{x^2 + y^2}$. The effective electric field is now along the radial direction, which leads to a response current

$$j_t(\mathbf{r}) = V_0 r \exp(-r^2/2\chi^2) / 2\hbar d\chi^2 \quad (11)$$

along the tangential direction. Such a current forms a close loop around the center that corresponds to a rotation of the superfluid (see Fig. 1). The velocity reaches its maximum at $r = \chi$ and then decreases on both sides. The direction of the rotation is determined by the chirality of the superfluid. This phenomenon can be observed by measuring the local velocity of the atoms using the Raman process. Remarkably, by applying a nonrotating laser beam, one can create a rotation of the condensate in a 2D $p_x + ip_y$ superfluid. Note that the total angular momentum is conserved in this process. In a $p_x + ip_y$ superfluid, each Cooper pair carries a unit of internal angular momentum. In our scheme, this angular momentum comes from the effective spin-orbit coupling for the atomic motion. The external nonrotating laser potential produces a density gradient of the Cooper pairs, leading to the redistribution of the angular momentum spatially. This redistribution of the angular momentum yields the tangential current peaking at $r = \chi$, and is the physical origin of the rotation of the condensate in a 2D $p_x + ip_y$ superfluid.

We emphasize that there is no antisymmetric transverse mass current in an s -wave or p_x -wave superfluid. Thus, the above experiments involving the transverse currents can

serve as clear diagnostic tests for the existence of the chiral superfluid. In the $p_x + ip_y$ superfluid, the time-reversal symmetry is broken, which leads to a nonzero Berry phase in the momentum space. The nonzero Berry phase, absent in the s - or p_x -wave superfluids, is the physical origin of the nonzero antisymmetric transverse mass current.

Finally, we note that our proposed method for observing the mass current is very general. It does not depend on the specific way to generate a $p_x + ip_y$ superfluid. Therefore, if a $p_x + ip_y$ superfluid can be generated using other methods (say, using a p -wave Feshbach resonance), our proposed diagnostic methods still apply.

In summary, we have proposed a concrete method to generate a chiral $p_x + ip_y$ cold atom fermionic superfluid by exploiting the well-established s -wave Feshbach resonance and the topological Berry phases, thereby circumventing the short lifetime issues of p -wave superfluids associated with p -wave Feshbach resonance. We have also proposed techniques for the direct observation of the chirality of the neutral $p_x + ip_y$ atomic superfluids in optical traps.

This work is supported by ARO-DARPA. We thank Professor Qian Niu for stimulating discussion.

-
- [1] C. Nayak *et al.*, Rev. Mod. Phys. **80**, 1083 (2008).
 - [2] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
 - [3] S. Tewari *et al.*, Phys. Rev. Lett. **98**, 010506 (2007).
 - [4] C. Zhang *et al.*, Phys. Rev. Lett. **99**, 220502 (2007).
 - [5] S. Tewari *et al.*, Phys. Rev. Lett. **100**, 027001 (2008).
 - [6] J. Xia *et al.*, Phys. Rev. Lett. **97**, 167002 (2006).
 - [7] S. Das Sarma *et al.*, Phys. Rev. B **73**, 220502(R) (2006).
 - [8] C. A. Regal *et al.*, Phys. Rev. Lett. **90**, 053201 (2003).
 - [9] C. H. Schunck *et al.*, Phys. Rev. A **71**, 045601 (2005).
 - [10] K. Günter *et al.*, Phys. Rev. Lett. **95**, 230401 (2005).
 - [11] J. P. Gaebler *et al.*, Phys. Rev. Lett. **98**, 200403 (2007).
 - [12] S. S. Botelho and C. A. R. Sa de Melo, J. Low Temp. Phys. **140**, 409 (2005).
 - [13] V. Gurarie *et al.*, Phys. Rev. Lett. **94**, 230403 (2005).
 - [14] C.-H. Cheng and S.-K. Yip, Phys. Rev. Lett. **95**, 070404 (2005).
 - [15] M. V. Berry, Proc. R. Soc. A **392**, 45 (1984).
 - [16] A. M. Dudarev *et al.*, Phys. Rev. Lett. **92**, 153005 (2004).
 - [17] J. Ruseckas *et al.*, Phys. Rev. Lett. **95**, 010404 (2005).
 - [18] S.-L. Zhu *et al.*, Phys. Rev. Lett. **97**, 240401 (2006).
 - [19] T. D. Stanescu *et al.*, Phys. Rev. Lett. **99**, 110403 (2007).
 - [20] J. Y. Vaishnav and C. W. Clark, Phys. Rev. Lett. **100**, 153002 (2008).
 - [21] T. P. Meyrath *et al.*, Phys. Rev. A **71**, 041604(R) (2005).
 - [22] I. B. Spielman *et al.*, Phys. Rev. Lett. **98**, 080404 (2007).
 - [23] J. Shi and Q. Niu, arXiv:0601531.
 - [24] S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 247 (1956).
 - [25] H. J. Metcalf and P. van der Straten, *Laser Cooling and Trapping* (Springer-Verlag, New York, 1999).
 - [26] R. M. Lutchyn *et al.*, Phys. Rev. B **77**, 144516 (2008)