

## Higher-Order Topological Corner States Induced by Gain and Loss

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Higher-order topological insulators and superconductors are topological phases that exhibit novel boundary states on corners or hinges. Recent experimental advances in controlling dissipation such as gain and loss in atomic and optical systems provide a powerful tool for exploring non-Hermitian topological phases. Here we show that higher-order topological corner states can emerge by introducing staggered on-site gain and loss to a Hermitian system in a trivial phase. For such a non-Hermitian system, we establish a general bulk-corner correspondence by developing a biorthogonal nested-Wilson-loop and edge-polarization theory, which can be applied to a wide class of non-Hermitian systems with higher-order topological orders. The theory gives rise to topological invariants characterizing the non-Hermitian topological multipole moments (i.e., corner states) that are protected by reflection or chiral symmetry. Such gain- and loss-induced higher-order topological corner states can be experimentally realized using photons in coupled cavities or cold atoms in optical lattices.

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*Introduction.*—Topological states of matter [1–5] have been widely studied in various systems ranging from solid-state [6–8], over cold atomic [9–17] to photonic [18–24] and acoustic [25–30] systems. The states are indexed by the bulk topological invariants that determine the boundary physics with lower dimensions. Recently, the concept has been generalized to higher-order topological insulators or superconductors with novel boundary states on corners or hinges [31–51]. Different from conventional first-order topological states, the  $d$ -dimensional  $n$ th-order topological states can host  $(d - n)$ -dimensional gapless boundary states. The experimental realizations of such interesting higher-order topological states in photonic [38–41] and electrical circuit [42,43] systems further enlighten the research of these novel topological matters.

Meanwhile, the search for topological states of matter has also turned to open quantum systems characterized by non-Hermitian Hamiltonians [52], which exhibit a rich variety of unique properties without Hermitian counterparts [53]. States modeled by non-Hermitian Hamiltonians appear in systems such as photonic structures with loss or gain [54–65], and cold atomic systems or solid-state materials with finite (quasi-) particle lifetime [66–73]. The eigenvalues are generally complex, and the right and left eigenstates, satisfying biorthonormality constraints, are no longer equivalent to each other (neither of them forms an orthogonal basis). Moreover, more than one right eigenstate can coalesce at exceptional points [71]. Such unique properties lead to a rich variety of interesting topological phenomena (e.g., the non-Hermitian skin effects, exceptional rings, bulk Fermi arcs, etc.), with bulk-boundary correspondence very different from the Hermitian systems [74–92].

The effects of non-Hermiticity on higher-order topological physics have been considered recently in a few works

[93–97], where the non-Hermiticity is induced by asymmetric tunnelings, leading to the observation of interesting phenomena such as higher-order skin effect [93] and biorthogonal bulk polarization [97]. Nevertheless, a general bulk-corner correspondence of the non-Hermitian higher-order topological states is still elusive. In addition, compared to asymmetric tunnelings, a simpler and more tunable way for introducing non-Hermiticity in photonic and atomic experiments is to control the on-site particle dissipations directly. Therefore two natural questions arise: (i) Can higher-order topological states be induced by simply controlling the on-site gain or loss? (ii) Is there a general bulk-corner correspondence for the non-Hermitian higher-order topological states?

In this Letter, we address these two important questions by considering a two-dimensional (2D) lattice model with staggered on-site particle gain and loss. Our main results are as follows: (i) The non-Hermitian particle gain and loss can drive the system from a trivial phase to a second-order topological phase with the emergence of four degenerate corner states. (ii) We develop the biorthogonal nested-Wilson-loop and edge-polarization approach, which gives rise to bulk topological invariants responsible for the gapless corner states. The topological invariants are protected by reflection or chiral symmetries. In the presence of additional  $C_4$  rotation symmetry, the topology can also be characterized by a quantized biorthogonal winding number. (iii) Although we focus on the 2D reflection-symmetric case, our model and the bulk-corner correspondence can be generalized to study  $d$ -dimensional  $d$ th-order non-Hermitian topological states with either reflection or chiral symmetries. (iv) Simple experimental schemes based on photons in coupled cavities and cold atoms in optical lattices are proposed. Our system only relies on the manipulation of on-site particle gain and loss, and is ready for experimental exploration.

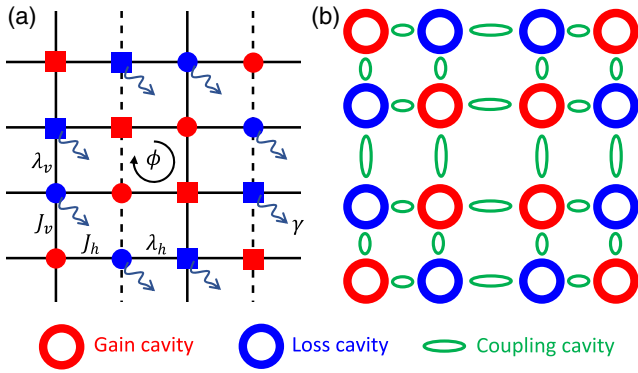


FIG. 1. (a) Lattice representation of the non-Hermitian model in Eq. (1). All sites in blue (red) have particle loss (gain) with a rate  $\gamma$ .  $\phi$  is the magnetic flux for each plaquette, and  $J_{h,v}$  ( $\lambda_{h,v}$ ) are the tunneling amplitudes between sites in different color (shape) along the horizontal and vertical directions, respectively. (b) Experimental implementation of the lattice model in (a) using coupled arrays of microring cavities.

*The model.*—We consider a 2D lattice model with staggered tunnelings along both horizontal and vertical directions, as shown in Fig. 1(a). There is an effective magnetic flux  $\phi = \pi$  for each plaquette, which appears as the tunneling phases on the dashed lines. The non-Hermiticity is introduced by the particle loss (gain) on all blue (red) lattice sites. We choose 16 orbitals in Fig. 1(a) as our unit cell with horizontal and vertical primitive-lattice vectors. The Hamiltonian reads

$$\begin{aligned}
 H(\mathbf{k}) = & J_h \sigma_h^x + J_v \sigma_v^x \sigma_h^\phi + i\gamma \sigma_h^z \sigma_v^z \tau_h^z \tau_v^z \\
 & + \lambda_h (\tau_h^- \sigma_h^+ + e^{-ik_x} \tau_h^- \sigma_h^- + \text{H.c.}) \\
 & + \lambda_v \sigma_h^\phi (\tau_v^- \sigma_v^+ + e^{-ik_y} \tau_v^- \sigma_v^- + \text{H.c.}), \quad (1)
 \end{aligned}$$

where  $J_{h,v} > 0$  ( $\lambda_{h,v} > 0$ ) are the nearest-neighbor tunneling amplitudes between red and blue (circle and square) sites,  $\sigma_{h,v}$  ( $\tau_{h,v}$ ) are the Pauli matrices for the degrees of freedom spanned by red and blue (circle and square) sites, and  $h, v$  represent the horizontal and vertical directions, respectively.  $\sigma_{h,v}^\phi = \sigma_{h,v}^z$  for  $\phi = \pi$ . The gain or loss rate  $\gamma$  in Eq. (1) is positive since the blue sites are lossy. Alternatively, we may consider a different gain or loss configuration with gain (loss) on blue (red) sites, which simply changes  $\gamma$  to negative. In experiments, the Hamiltonian can be realized using cold atoms in optical lattices or photons in coupled cavities [98]. Figure 1(b) is an example based on arrays of coupled microring cavities, where the coupling amplitude and phase between neighbor cavities, and the photon gain and loss for each cavity, can be controlled independently [41,62].

*Corner states.*—For simplicity, we assume  $J_h = J_v \equiv J$  throughout this Letter, and the physics for  $J_h \neq J_v$  is similar. The system has 16 bands [98], which appear in pairs  $E(\mathbf{k}) = -E^*(\mathbf{k})$  due to the pseudoanti-Hermiticity

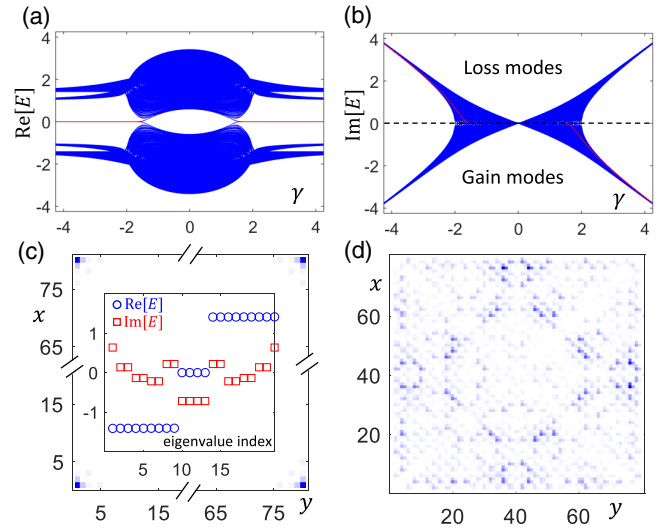


FIG. 2. [(a) and (b)] Energy spectra of the non-Hermitian Hamiltonian Eq. (1) with open boundaries in both directions. The bulk energy gap closes at  $|\gamma| = \gamma_c$ , where a topological phase transition occurs and in-gap corner states (red curves with fourfold degeneracy) emerge at  $|\gamma| > \gamma_c$ . (c) Typical density distributions  $|\Psi_{\text{corner}}^R(x,y)|^2$  of the four corner states, with  $|\Psi_{\text{corner}}^R(x,y)\rangle$  being the right eigenstate. The inset shows the corresponding eigenenergies around the four corner states with  $\text{Re}[E] = 0$ . (d) Typical density distributions of the bulk states.  $\gamma = 2$  in (c) and (d). Common parameters are system size  $N_h = N_v = 20$  (unit cells),  $J = \sqrt{2}$ , and  $\lambda_v = 1$  (leading to  $\gamma_c = \sqrt{2}$ ). We set  $\lambda_h = 1$  as the energy unit.

$\eta H \eta = -H^\dagger$  with  $\eta = \sigma_h^z \sigma_v^z \tau_h^z \tau_v^z$ . We are interested in the half-filling gap around  $\text{Re}[E] = 0$ . We focus on the region  $\lambda_{h(v)} \leq J$  (the system stays in the trivial insulating or metal phase at the Hermitian limit  $\gamma = 0$  [32,33]), and show that the second-order topological corner states can be induced solely by non-Hermitian gain or loss.

In Figs. 2(a) and 2(b), we plot the energy spectrum as a function of  $\gamma$ , with open boundaries along both directions. Effectively, the particle loss reduces the tunnelings between gain and loss sites, while the tunnelings between two loss (gain) sites are not affected. We see that as  $|\gamma|$  increases, the bulk gap closes and reopens (the small derivation is the finite size effect) at a critical point  $\gamma_c$ , leading to a topological phase transition with the emergency of four in-gap states. The typical density distributions of these in-gap states are shown in Fig. 2(c), which are well localized at four corners. We emphasize that our system does not suffer from the non-Hermitian skin effects due to the trivial eigenenergy vorticity [81]  $\oint \partial_{\mathbf{k}} \text{Arg}[E(\mathbf{k})] d\mathbf{k} = 0$  for any loop in the momentum space; therefore it does not matter whether the right and/or left eigenstates are used to calculate the density distribution. As a result, the bulk states of  $H(\mathbf{k})$  do distribute in the bulk [see Fig. 2(d)], and the open-boundary bulk spectrum is the same as that for periodic boundaries. We set  $\lambda_h = \lambda_v$  in Fig. 2; therefore the system undergoes a bulk gap closing across the topological

phase transition due to the  $C_4$  symmetry [46]. In general, the second-order topology can be altered by the gap closing in either the bulk or edge spectrum, and the emergency of corner states does not require bulk energy gap closing for  $\lambda_h \neq \lambda_v$  [33,46], which is further illustrated.

*Topological invariants.*—For Hermitian systems, it was shown that the topology of the nested Wilson loop and edge polarization are responsible for the corner states [32,33]. Here we develop their non-Hermitian counterparts and show that the non-Hermitian corner states are originated from the topology of the generalized biorthogonal nested Wilson loops and edge polarizations. We consider a general Hamiltonian  $H(\mathbf{k})$  on a torus with periodic boundaries and define the biorthogonal Wilson loop operator as

$$W_{h,\mathbf{k}} = \mathcal{P} \exp \left[ i \int_{k_x}^{k_x+2\pi} A_h(k'_x, k_y) dk'_x \right], \quad (2)$$

where  $A_h(\mathbf{k}) = -i \langle u_{n,\mathbf{k}}^L | \partial_{k_x} | u_{m,\mathbf{k}}^R \rangle$  is the biorthogonal non-Abelian Berry connection in the horizontal direction,  $|u_{m,\mathbf{k}}^{R,L}\rangle$  are the  $m$ th occupied right and left Bloch eigenstates satisfying  $H(\mathbf{k})|u_{m,\mathbf{k}}^R\rangle = E_m(\mathbf{k})|u_{m,\mathbf{k}}^R\rangle$ ,  $H^\dagger(\mathbf{k})|u_{m,\mathbf{k}}^L\rangle = E_m^*(\mathbf{k})|u_{m,\mathbf{k}}^L\rangle$  and  $\langle u_{n,\mathbf{k}}^L | u_{m,\mathbf{k}}^R \rangle = \delta_{n,m}$ , and  $\mathcal{P}$  is the path-ordering operator. Different from the Hermitian case [32],  $W_{h,\mathbf{k}}$  may no longer be a unitary operator, and leads to a non-Hermitian Wannier Hamiltonian  $H_{W_h}(\mathbf{k}) = -(i/2\pi) \log W_{h,\mathbf{k}}$ , which also has different left and right eigenstates, that is,  $H_{W_h}(\mathbf{k})|e_{h,j,\mathbf{k}}^R\rangle = \varepsilon_{h,j,k_y}|e_{h,j,\mathbf{k}}^R\rangle$ ,  $H_{W_h}^\dagger(\mathbf{k})|e_{h,j,\mathbf{k}}^L\rangle = \varepsilon_{h,j,k_y}^*|e_{h,j,\mathbf{k}}^L\rangle$  with  $\langle e_{h,j,\mathbf{k}}^L | e_{h,j,\mathbf{k}}^R \rangle = \delta_{j,j'}$ , and  $j$  being the Wannier band index. The non-Hermitian Wannier bands (independent from  $k_x$ ), which obey the identification  $\text{Re}[\varepsilon_{h,j,k_y}] \equiv \text{Re}[\varepsilon_{h,j,k_y}^*] \pmod{1}$ , can carry topological invariants if they are gapped.

The biorthogonal vertical polarization for the Wannier-band sector  $\varepsilon_h$  can be defined as

$$p_v^{\varepsilon_h} = -\frac{i}{4\pi^2} \int dk_x \log \det[\tilde{W}_{h,\mathbf{k}}]. \quad (3)$$

Here  $\tilde{W}_{h,\mathbf{k}}$  is the biorthogonal nested Wilson loop along the vertical direction, which is defined on the Wannier sector  $\varepsilon_h$  with non-Hermitian Wannier-band basis  $|w_{h,j,\mathbf{k}}^{R(L)}\rangle = \sum_{m=1}^{N_{\text{occ}}} |u_{m,\mathbf{k}}^{R(L)}\rangle [e_{h,j,\mathbf{k}}^{R(L)}]_m$  ( $N_{\text{occ}}$  is the number of occupied energy bands and  $\langle w_{h,j,\mathbf{k}}^L | w_{h,j',\mathbf{k}}^R \rangle = \delta_{j,j'}$ ) [98]. Similarly, we can obtain the biorthogonal nested Wilson loop along the horizontal direction and the corresponding polarization  $p_h^{\varepsilon_v}$ . There would be corner states when  $p_{h,v}^{\varepsilon_{v,h}}$  are nontrivial.

On the other hand, even for trivial  $p_{h,v}^{\varepsilon_{v,h}}$ , one may still have corner states if the edge polarization is nontrivial [33]. For non-Hermitian systems, we should use the biorthogonal edge polarization, which is obtained by considering a cylindrical geometry and calculating the pseudo-one-dimensional biorthogonal Wannier values ( $\varepsilon_{h,j}$  or  $\varepsilon_{v,j}$ ) and polarization ( $p_h^{i_v}$  or  $p_v^{i_h}$  with  $i_v$  or  $i_h$  being the unit-cell

index along the open direction) along the periodic direction (horizontal or vertical) [98]. The second-order corner modes are characterized by the vanishing bulk polarization (i.e.,  $i_{v,h}$  away from 1 and  $N_{v,h}$ ), but quantized nonzero edge-localized polarization  $p_h^{\text{edge}}$  and/or  $p_v^{\text{edge}}$  (i.e.,  $i_{v,h}$  near 1 or  $N_{v,h}$ ) [98].

In general, higher-order topological phases are protected by symmetries [32,33]. We consider a Hamiltonian that respects either reflection symmetries  $M_h H(k_x, k_y) M_h^{-1} = H(-k_x, k_y)$  and  $M_v H(k_x, k_y) M_v^{-1} = H(k_x, -k_y)$ , or chiral (sublattice) symmetry  $\Xi H(k_x, k_y) \Xi^{-1} = -H(k_x, k_y)$ , with symmetry operators given by  $M_h$ ,  $M_v$  or  $\Xi$ . Since the biorthogonal Wannier bands or values (on a torus or cylinder) change the signs under reflection operation, they are either flat bands locked at 0 or  $\frac{1}{2}$ , or appear in  $\pm\varepsilon$  pairs for reflection-symmetric systems. The reflection symmetries also ensure the quantization of  $(p_h^{\varepsilon_v}, p_v^{\varepsilon_h})$  and  $(p_h^{\text{edge}}, p_v^{\text{edge}})$  with value 0 or  $\frac{1}{2}$ . Similar properties hold for the chiral-symmetric systems with non-Hermiticity induced by asymmetric tunneling [98].

The Wannier bands correspond to the position of the particle density cloud [32,33]. We focus on the Wannier sectors  $\in (0, \frac{1}{2})$  [or  $\in (\frac{1}{2}, 1)$ ], which are responsible for the edge topology and corner states. Based on the Wannier-sector and edge polarizations, we define two topological invariants:  $Q_1 = 4p_v^{\varepsilon_h} p_h^{\varepsilon_v} \pmod{2}$  [with  $\varepsilon_{v,h}$  being the Wannier sector  $\in (0, \frac{1}{2})$ ] and  $Q_2 = 2(p_h^{\text{edge}} + p_v^{\text{edge}}) \pmod{2}$ . For the topological phase, we have either  $Q_1 = 1$  or  $Q_2 = 1$ ; while for the trivial phase, we have both  $Q_1 = 0$  and  $Q_2 = 0$ . The above bulk-corner correspondence can apply to any non-Hermitian systems with reflection or chiral symmetries, and is reduced to the normal nested-Wilson-loop and edge-polarization theory [32] in the Hermitian limit.

*Phase diagram.*—As an example, we study the phase diagram of the model in Fig. 1 based on the biorthogonal topological invariants. The corresponding Hamiltonian satisfies reflection symmetries with  $M_h = \sigma_v^y \sigma_h^x \tau_h^x$  and  $M_v = \sigma_v^x \tau_v^x$ . It also possesses the rotational symmetry  $C_4 H(k_x, k_y) C_4^{-1} = H(k_y, -k_x)$  if  $\lambda_h = \lambda_v$ , where  $C_4 = C_\tau \otimes C_\sigma$  with

$$C_\tau = \frac{1}{2} \sum_{s \neq \bar{s}} \left[ \tau_s^+ \left( \frac{1 - \tau_{\bar{s}}^z}{2} \right) + \tau_s^- \left( \frac{1 + \tau_{\bar{s}}^z}{2} \right) \right], \quad (4)$$

and  $C_\sigma$  has a similar expression with  $s, \bar{s} = \{h, v\}$ . In Fig. 3(a), we show the phase diagram in the  $\gamma - \lambda_v$  plane with  $J/\lambda_h = \sqrt{2}$ . The phase diagram is symmetric with respect to  $\gamma = 0$ , so we focus on  $\gamma \geq 0$ . The right and left parts of the phase diagram belong to topological and trivial phases, with their boundary given by the solid line. The trivial phase enlarges with the phase boundary shifting rightward as we increase  $J$ . There are two topological phases:  $T$ -I with  $(Q_1, Q_2) = (1, 0)$  and  $T$ -II with  $(Q_1, Q_2) = (0, 1)$ .

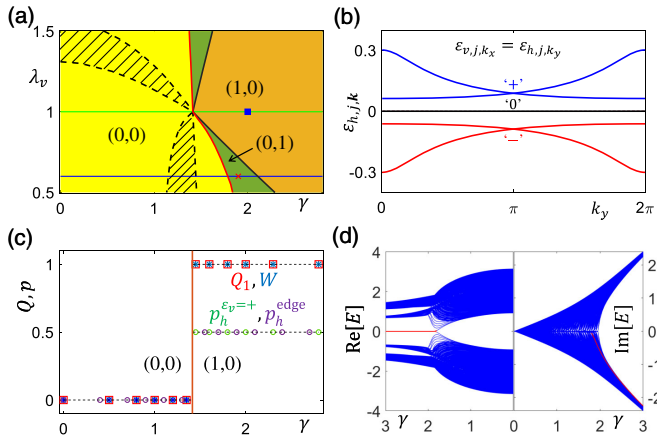


FIG. 3. (a) Phase diagram in the  $\gamma$ - $\lambda_v$  plane for  $J = \sqrt{2}\lambda_h$ , with one trivial phase [yellow area with  $(Q_1, Q_2) = (0, 0)$ ] and two topological phases [ $T$ -I, orange area with  $(1, 0)$ ; and  $T$ -II, green area with  $(0, 1)$ ]. The patterned region has a vanishing Wannier-band gap [98]. (b) Wannier-band structures with  $\lambda_v = \lambda_h$  and  $\gamma = 2$  [the blue square in (a)]. The imaginary parts are locked at 0. (c) Wannier-sector (green circles) and edge (purple circles) polarizations as well as topological invariant  $Q_1$  (red squares) and winding number  $W$  (blue stars), with  $\lambda_v = \lambda_h$  [the thin green line in (a)]. (d) Complex energy spectra with open boundaries [ $N_h = N_v = 20$  (unit cells)] and  $\lambda_v = 0.6\lambda_h$  [the thin blue line in (a)]. The bulk gap persists upon the phase transition. We set  $\lambda_h = 1$  as the energy unit.

We first consider the  $C_4$  symmetric case for  $\lambda_v = \lambda_h$ , with the open-boundary spectra shown in Fig. 2(a). The typical Wannier bands for the Hamiltonian Eq. (1) with periodic boundaries are shown in Fig. 3(b). There are eight Wannier bands, with four located around  $\varepsilon = 0$ , two at  $0 < \text{Re}[\varepsilon] < \frac{1}{2}$  and two at  $0 > \text{Re}[\varepsilon] > -\frac{1}{2}$ , forming three Wannier sectors labeled by  $0, \pm$ , as shown in Fig. 3(b). Only the  $\pm$  Wannier sectors are responsible for the edge topology and corner states. In fact, the  $0$  Wannier sector is trivial in the whole parameter space and the  $\pm$ -Wannier sectors always have the same topology. Because of the  $C_4$  symmetry, we have  $p_h^{\varepsilon_v=\pm} = p_v^{\varepsilon_h=\pm}$  and  $p_h^{\text{edge}} = p_v^{\text{edge}}$ , all of which jump from  $0$  to  $\frac{1}{2}$  across the phase transition as  $\gamma$  increases [see Fig. 3(c)]. We point out that, with  $C_4$  symmetry, the topology can also be characterized by the biorthogonal winding number  $W$  along the high-symmetry line  $k_x = k_y$  in the reflection-rotation ( $C_4M_h$ ) subspace [98].

For  $\lambda_v \neq \lambda_h$ , the bulk energy gap persists [see Fig. 3(d)], and the phase transitions are driven by gap closes or reopenings in the edge spectra and the Wannier bands, which lead to polarization jumps. In the following, we focus on  $\lambda_v < \lambda_h$  without loss of generality, and show how the topological invariants and phases change as we increase  $\gamma$ , as shown in Figs. 4(a) and 4(b). (i) First, the vertical Wannier bands  $\varepsilon_{v,j,k}$  close the gap between  $0$  and  $\pm$  sectors in the patterned region in Fig. 3(a) [98]. Further increasing  $\gamma$  reopens the gap and leads to the jump of  $p_h^{\varepsilon_v=\pm}$  from

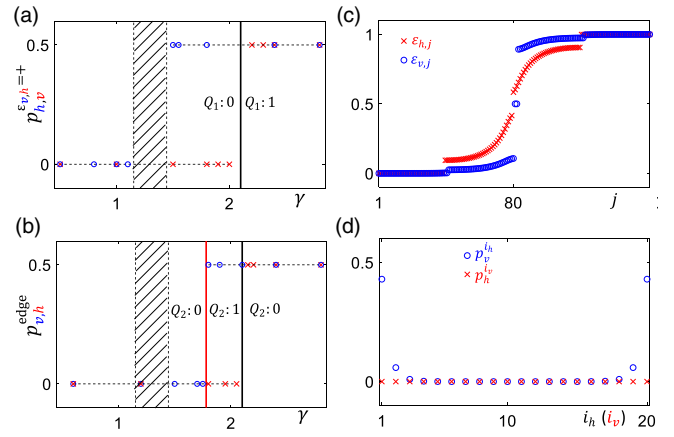


FIG. 4. [(a) and (b)] The horizontal (blue circles) and vertical (red crosses) polarizations for  $\lambda_v = 0.6\lambda_h$  [along the thin blue line in Fig. 3(a)]. The polarization  $p_h^{\varepsilon_v}$  is ill defined in the patterned region due to the vanishing gap between Wannier sectors. The phase boundaries are given by the solid lines. [(c) and (d)] The Wannier values ( $\varepsilon_{h,j}$ ,  $\varepsilon_{v,j}$ ) and edge-polarization distributions ( $p_h^{i_v}$ ,  $p_v^{i_h}$ ) for the cylindrical geometry with  $\lambda_v = 0.6\lambda_h$  and  $\gamma = 1.9$  [indicated by the red cross in Fig. 3(a)]. Blue circles (red crosses) are the results for open boundary along the horizontal (vertical) direction. The in-gap Wannier values at  $\frac{1}{2}$  are responsible for the edge polarization. Other parameters are the same as in Fig. 3.

$0$  to  $\frac{1}{2}$ . (ii) Then, the gap for the edge spectra closes and reopens on the red solid line, where  $p_v^{\text{edge}}$  jumps from  $0$  to  $\frac{1}{2}$  and the system enters the  $T$ -II phase with the emergence of corner states. Shown in Figs. 4(c) and 4(d) are the Wannier values ( $\varepsilon_{h,j}$  and  $\varepsilon_{v,j}$ ) and edge-polarization distribution ( $p_h^{i_v}$  and  $p_v^{i_h}$ ) for the phase  $T$ -II on a cylinder. (iii) Finally,  $\varepsilon_{h,j,k}$  close the gap between  $+$  and  $-$  sectors on the black solid line, where both  $p_v^{\varepsilon_h=\pm}$  and  $p_h^{\text{edge}}$  jump from  $0$  to  $\frac{1}{2}$ , and we reach the  $T$ -I phase. Both  $T$ -I and  $T$ -II phases support corner states, and they are distinguished by the edge topology [98]. The  $T$ -II phase region shrinks to  $0$  as  $\lambda_v$  approaches  $\lambda_h$ , where all edge and Wannier-sector polarizations jump at the same  $\gamma$  due to the  $C_4$  symmetry. These phenomena are very different from the Hermitian case. Especially, one can only have the topological phase  $T$ -I for the Hermitian limit, where all edge polarizations must vanish as long as  $Q_1 = 0$  [33]. The appearance of phase  $T$ -II is a result of the interplay between the non-Hermiticity and the  $C_4$  symmetry breaking [98].

*Discussion.*—It is possible to generalize our study by considering different flux configurations. As a simple example, one may consider  $\phi = 0$  and set  $\sigma_{h,v}^{\phi} = \sigma_{h,v}^0$  in the Hamiltonian Eq. (1). For such a zero flux model, the Hermitian part is a gapless metal when  $|\lambda_h - \lambda_v| \leq 2J$ . The gain and loss term effectively reduces the tunneling  $J$  and can open a topological gap with in-gap corner states [98]. Moreover, it is straightforward to generalize our non-Hermitian model and bulk-corner correspondence to

higher-dimensional systems (e.g., a 3D system supporting third-order topological phases with quantized octupole moment) [98]. Finally, we consider a general asymmetric-tunneling model (without on-site gain or loss) as an example of chiral-symmetric systems, and confirm the bulk-corner correspondence numerically [98]. The asymmetric tunnelings break both the Hermiticity and reflection symmetries (other symmetries like  $C_4$  rotation or reflection-rotation  $C_4M_h$  are also broken). As we increase the strength of the non-Hermiticity (i.e., asymmetry), the system can transform from the trivial phase to the second-order topological phase with zero-energy modes at four corners, which are characterized by the nontrivial topology of the biorthogonal nested Wilson loops [98].

*Conclusion.*—In summary, we propose a scheme to realize non-Hermitian higher-order topological insulators by simply controlling the on-site gain or loss, and show that the non-Hermitian corner states are characterized by the bulk topology in the form of biorthogonal nested Wilson loops or edge polarizations. The generalized bulk-corner correspondence may work for a wide class of non-Hermitian  $d$ -dimensional  $d$ -order topological systems with reflection or chiral symmetries. The proposed model can be realized easily in experiments. Our work offers a tunable method for manipulating corner states through dissipation control, and paves the way for the study of various non-Hermiticity induced higher-order topological states of matter and the classifications of them.

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