

Fidelity of a Bose–Einstein condensates

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Abstract

We investigate fidelity for the quantum evolution of a Bose–Einstein condensate and reveal its general property with a simple model. We find the fidelity decay with time in various ways depending on the form of initial states as well as on mean-field dynamics. When the initial state is a coherent state, the fidelity decays with time in the ways of exponential, Gaussian, and power-law, having a close relation to the classical mean-field dynamics. With the initial state prepared as a maximally entangled state, we find the behavior of fidelity has no classical correspondence and observe a novel behavior of the fidelity: periodic revival, where the period is inversely proportional to the number of bosons and the perturbation strength. An experimental observation of the fidelity decay is suggested.

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Instability issue of Bose–Einstein condensation (BEC) has been constantly addressed for its crucial role in control, manipulation and even future's application of this newly formed matter. Dynamical instability [1], Landau or superfluid instability [2], modulation instability [3] and quantum fluctuation instability [4] have been discussed thoroughly. It is found that instability will break the coherence among the atoms and therefore lead to the collapse of BEC [5].

However, an important issue is still missing, namely, the sensitivity of the quantum evolution of a BEC with respect to a perturbation from outer environment. This instability is distinguished from the instability mentioned above in that the perturbation here is from outer environment rather than from the inner of system. It can be depicted by the so-called fidelity, or the Loschmidt echo, defined as the overlap of two states obtained

by evolving the same initial state under two slightly different (perturbed and unperturbed) Hamiltonians [6–9]. This issue is very essential for coherent manipulation of BEC as well as for future's application of BEC to quantum information and quantum computation [10,11].

In this Letter, we discuss this issue by considering a two-component BEC trapped in a harmonic potential [12], subject to a periodic coupling (successive kicks) between the two components. This is a rather general model containing rich dynamical behavior as we show later, with a constant coupling it is a BEC system proposed recently to generate entangled state for quantum computation [11]. Taking this simple model for example, we investigate the new instability of BEC and reveal its general property. We also show that the fidelity instability (fast decay) may lead to a fadeaway of the inference pattern of recent experiment [13,14].

The two internal states of the BEC are coupled by a near resonant pulsed radiation field [12]. Total density and mean phase remain constant during the condensate evolution, then the

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Hamiltonian describing the transition between the two internal states reads [15]

$$\hat{H} = \mu(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) + g(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)^2 + K\delta_T(t)(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (1)$$

where K is the coupling strength between the two internal states, g is the interaction, and μ is the difference between the chemical potentials of two components. $\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2$ and \hat{a}_2^\dagger are boson annihilation and creation operators for the two components, respectively. $\delta_T(t) = \sum_n \delta(t - nT)$ means that the radiation field is only turned on at certain discrete moments, i.e., integral multiples of the period T .

The above model is a very simple model used to demonstrate our theory. In realistic experiments, the decoherence effects always exist. Generally, decoherence originates in the coupling to a bath of unobserved degree of freedom, or the interparticle entanglement process [16,17]. The main source of decoherence in a BEC is the thermal cloud of particles surrounding the condensate. Thermal particles scattering off the condensate will cause phase diffusion at a rate Γ proportional to the thermal cloud temperature. For internal states not entangled with the condensate spatial state, Γ maybe as low as 10^{-5} Hz under the coldest experimental condition [18].

Writing the above Hamiltonian in terms of the angular momentum operators [18],

$$\hat{L}_x = \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1}{2}, \quad \hat{L}_y = \frac{\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1}{2i},$$

$$\hat{L}_z = \frac{\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2}{2},$$

we have $\hat{H} = \mu\hat{L}_z + g\hat{L}_z^2 + K\delta_T(t)\hat{L}_x$. The Floquet operator depicting the quantum evolution in one period takes the following form:

$$\hat{U} = \exp[-i(\mu\hat{L}_z + g\hat{L}_z^2)T] \exp(-iK\hat{L}_x). \quad (2)$$

The Hilbert space is spanned by the eigenstates of $\hat{L}_z, |l\rangle$, with $l = -L, -L + 1, \dots, L$, where $L = N/2$ and N is the total number of atoms. In the above expression and henceforth, the Planck constant is set to unit.

The outer perturbation is mimicked by a small change on the interaction parameter or on the coupling strength, etc. Without losing generality, here we suppose that perturbation is on coupling, i.e., a small perturbation leads to the change of the coupling strength like $K \rightarrow K + \varepsilon$, the corresponding evolution operator is denoted by \hat{U}_ε . To investigate the influence of the perturbation on the quantum evolution, we need trace the temporal evolution of the fidelity function $M(t) = |m(t)|^2$, here $m(t)$ is the Loschmidt echo defined as

$$m(t = nT) = \langle \Phi_0 | (\hat{U}_\varepsilon^\dagger)^n \circ (\hat{U})^n | \Phi_0 \rangle, \quad (3)$$

where $|\Phi_0\rangle$ is the initial state and the fast decay of the fidelity means the rapid lose of the information during quantum evolution in the presence of perturbation.

We first set the initial state as a coherent state, $|\Phi_0\rangle = e^{\alpha^* \hat{L}_+ - \alpha \hat{L}_-} | -L \rangle$, with $\alpha = \frac{\pi - \theta}{2} e^{-i\phi}$. The system parameters

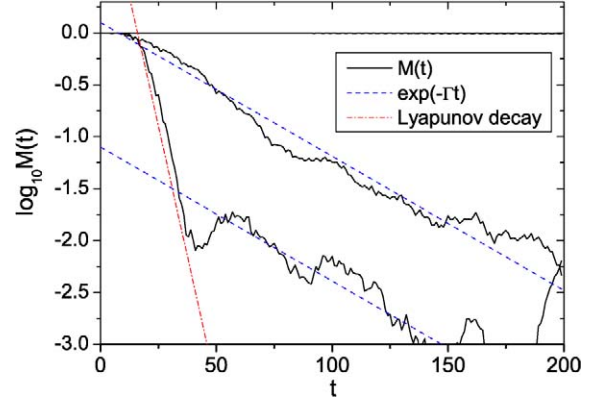


Fig. 1. Fidelity decay in the mixed system whose classical phase space structure is shown in left panel of Fig. 2. $L = 500$ and $\varepsilon = 6 \times 10^{-4}$. The non-decaying solid line is the fidelity of an initial coherent state lying within the largest regular region. The other two solid curves correspond to fidelity of two initial coherent states lying in the chaotic region of the classical system. One of them has an exponential decay with $\Gamma = 0.03$. Unexpectedly, the other one first has a fast Lyapunov decay $e^{-\lambda t}$, with λ being the Lyapunov exponent, then follows the exponential decay as the first one. Detail refer to the text.

are set as $\mu = T = 1$. With three different coherent states of the parameters $(\phi, \cos \theta) = (0.25, 0.04), (4.97, -0.2),$ and $(4.91, -0.9)$, respectively, we then calculate the temporal behavior of the fidelity numerically by solving the operator equations (2), (3) with the standard FFT method. The results are shown in Fig. 1. We find fidelity decay depends strongly on the initial parameters (θ, ϕ) : it may have no decay up to $t = 200$ as in the first case, or decays much faster in an exponential way as in the latter two cases.

To understand the above phenomena, we need retrospect the classical limit of the above quantum system. The effective Planck constant of the system $\hbar_{\text{eff}} = 1/L$. In the limit $N \rightarrow \infty$, it describes a classical spin on a Bloch sphere with $S_i = \frac{1}{L} \langle \hat{L}_i \rangle$ ($i = x, y, z$). The classical Hamiltonian takes the form, $H = \mu S_z + g_c S_z^2 + K\delta_T(t) S_x$, and the equations $\dot{S}_i = [S_i, H]_{\text{cl}}$ ($i = x, y, z$), where $g_c = gL$. This classical equations is just the mean field Gross–Pitaevskii equation for the BEC system. They can be solved analytically: the free evolution between two consecutive kicks corresponds to a rotation around S_z axis with the angle $(\mu + 2g_c S_z)T$, and the periodic kicks added at times nT give rotation around the S_x axis with the angle K . We then plot the classical orbits in left panel of Fig. 2. It shows one big island and four small islands. Inside the islands motions are stable or quasi-periodic, outside the islands are the chaotic points indicating the unstable motions. The chaotic motion is characterized by an exponential magnification of initial deviation having a positive Lyapunov exponent,

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{|\delta x(t)|}{|\delta x(0)|},$$

here $|\delta x(t)|$ denoting distance in phase space.

Corresponding to the coherent state whose fidelity almost has no decay with time, we find its location is inside the large island; the other two coherent states that have a fast fidelity decay lie in the chaotic region. To provide more evidence for this observation, we make extensive calculations on the fidelity of the

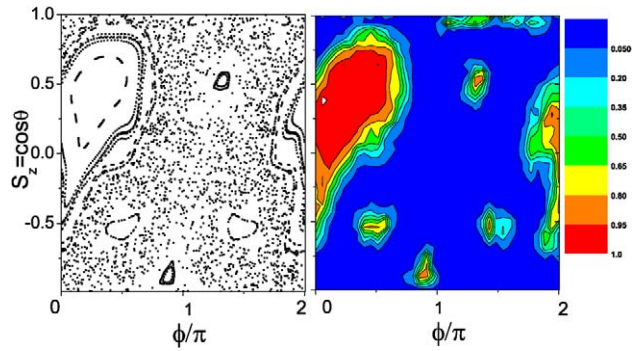


Fig. 2. Left panel: stroboscopic plots of the orbits for $g_c = 1$, $K = 2$, where x -axis ϕ is the azimuthal angle. It shows one big island and four small islands. Inside the islands motions are stable, outside the islands motions are mainly unstable or chaotic. Right panel: contour plot of the fidelity $M(t)$ at $t = 200$, for $K = 2$, $g_c = 1$, $L = 500$, and $\epsilon = 0.0006$. The initial quantum states are coherent states with corresponding (S_z, ϕ) .

coherent states with different initial positions. The right panel of Fig. 2 shows the contour plot of their fidelity at time $t = 200$. Its structure is quite similar to the phase space of the classical system. This similarity can be understood from different dispersion behavior for the coherent state wavepackets started from regular region and chaotic region, respectively. Compared to that in regular region, in the chaotic region the wavepackets disperse much faster and soon become extended (for example) in ϕ -representation. We then imagine that the overlap of the two states governed by slightly different (perturbed and unperturbed) Hamiltonians may decay fast with time. It means that the dynamical instability regime of the classical system usually corresponds to the low fidelity regime of the quantum system. On the other hand, inside the islands (the large or small) where the classical motions are dynamically stable with zero Lyapunov exponent, the fidelity shows different behavior: the fidelity in the large island is higher than that in the small islands. In the connected chaotic region where the Lyapunov exponent is a certain positive value, the fidelity may show different behavior as in Fig. 1. These facts elucidate that fidelity contains more information about the system under a perturbation and therefore is a more general quantity to describe the stability of the BEC.

The system parameters g and K dramatically influence the phase space and the fidelity of the system. We trace the Lyapunov exponent of orbits with initial points randomly scattered in the whole phase space and calculate fraction of the chaotic orbits. Its dependence on system parameters (g_c, K) is shown in Fig. 3. We find that the integrable cases (with few fraction of chaotic orbits in phase space) mainly concentrate on the vertical line where the interaction strength vanishes, and on the horizontal lines where the coupling strength is a multiple of π . We then make extensive numerical calculations on fidelity decay for the cases that the classical phase space is full of chaotic orbits (e.g., $K = 2$, $g_c = 4$) or full of near-integrable orbits (e.g., $K = 2$, $g_c = 0.2$), respectively. Basically, the fidelity for the fully chaotic case is similar to the chaotic region of the mixed case discussed above (i.e., $K = 2$, $g_c = 1$), showing a fast decay, whereas the fidelity for the near-integrable case shows a

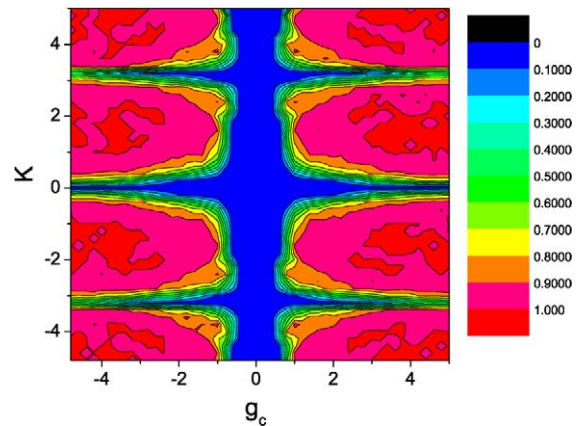


Fig. 3. Contour plot of the fraction of chaotic orbits in phase space, with respect to system parameters.

slow decay in principle like that in stable islands of the mixed case. Taking the near-integrable case for example, we find a Gaussian decay for the fidelity of single initial coherent state, with the decaying rate depending on the initial position of the coherent state. However, after averaging over the whole phase space, we find that the fidelity decay can be well fitted by an inverse power law $1/t$. Details will be presented elsewhere [19].

In the above discussions we set the initial state as coherent state. Actually, the quantum degenerate atomic gas is a fertile ground for exploring quantum information and quantum computation, where entanglement is crucial. Discussing fidelity of the entangled state is of great interest and of practical significance. The GHZ state for up to a few tens of bosons through entangling their internal states is proposed and tested realizable in [11]. In following discussions we assume a maximum entangled state or N -GHZ (Greenberger–Horne–Zeilinger) state is generated initially, which can be written as follows for our N -bosons system,

$$|\text{GHZ}\rangle_N = \frac{1}{\sqrt{2}} \left(\frac{\hat{a}_1^{\dagger N}}{\sqrt{N!}} + \frac{\hat{a}_2^{\dagger N}}{\sqrt{N!}} \right) |0\rangle = \frac{1}{\sqrt{2}} (|{-L}\rangle + |L\rangle), \quad (4)$$

we want to discuss the fidelity for its quantum evolution.

Before present our numerical results, we make some simple deductions and give a theoretical prediction for the fidelity behavior of the entangled state. Suppose the perturbation is very small, we can ignore the term like $\langle L | (\hat{U}_\epsilon^\dagger)^n \circ (\hat{U})^n |{-L}\rangle$. Then the Loschmidt echo for a GHZ state may simply be expressed as the sum of the echoes of the Fock states $|{-L}\rangle$ and $|L\rangle$, namely,

$$m_{|\text{GHZ}\rangle}(t) \simeq \frac{1}{2} m_{|L\rangle}(t) + \frac{1}{2} m_{|{-L}\rangle}(t). \quad (5)$$

If the difference between the echo functions of the two Fock states ($||m_{|L\rangle}(t)| - m_{|{-L}\rangle}(t)||$) is large, the interference terms in the fidelity expression of the GHZ state will be negligible, the fidelity of the GHZ state will approximate to the fidelity of one Fock state that have large absolute echo, showing a monotonous decrease with time. On the other hand, if the echoes of the two Fock states are comparable, the fidelity of the GHZ state will show an oscillation between minimum $\frac{1}{4} ||m_{|L\rangle}(t)| - m_{|{-L}\rangle}(t)||^2$ and maximum $\frac{1}{4} ||m_{|L\rangle}(t)| + m_{|{-L}\rangle}(t)||^2$. The pe-

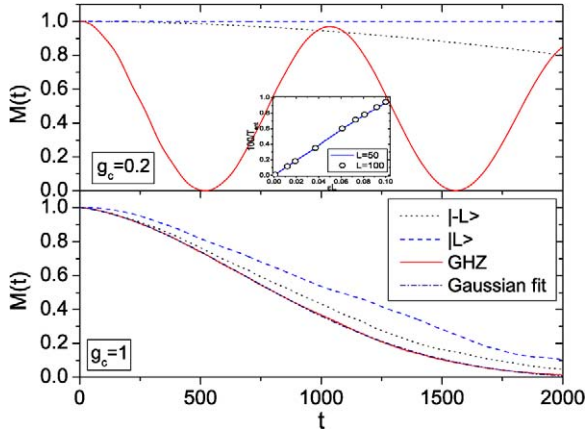


Fig. 4. Fidelity for the two Fock states and the GHZ entangled state, at $L = 500$, $\epsilon = 2 \times 10^{-5}$ and $K = 2$. The upper panel is with $g_c = 0.2$. Inset of upper panel: $100/T_{\text{ent}}$ vs ϵL for ϵL from 0.001 to 0.1, with $L = 50$ (solid line) and $L = 100$ (circles), showing the linear dependence of T_{ent} on $1/\epsilon L$. The lower panel is for $g_c = 1$, in which the fidelity of the GHZ state can be well fitted by a Gaussian decay.

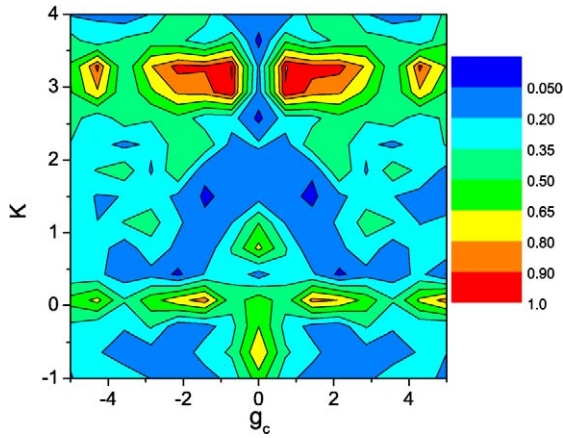


Fig. 5. Contour plot of the fidelity $M(t)$ at $t = 1500$, with respect to the parameters g_c, K . The initial quantum states are GHZ entangled states.

riod of this oscillation is determined by the relative phase between the echoes of the two Fock states. We have employed the semiclassical approach to give an estimation to the period T_{ent} of the oscillation, and find it inversely proportional to the number of the bosons and the perturbation strength, i.e., $T_{\text{ent}} \propto 1/(\epsilon L)$ [19].

Our numerical simulations prove the above theoretical predictions, as shown in Fig. 4. The periodic oscillation in upper panel of Fig. 4 is a quite interesting phenomenon, it indicates the fidelity already decayed very low can revive after a certain time duration. This type of fidelity behavior is a unique property for the entangled state.

To have a knowledge of the global situation of fidelity decay for the GHZ states with respect to the system’s parameters, we calculate the fidelity for a wide range of parameters as shown in Fig. 5. Compared with the contour plot of classical dynamics Fig. 3, we find no clear evidence for the correspondence.

In conclusions, we investigate the instability of a BEC under an external perturbation and reveal its general property.

This new instability characterized by the fast fidelity decay may lead to observable phenomenon. Let us consider BECs (e.g., ^{87}Rb) that are optically cooled and trapped and are then transferred into a double-well potential. The double-well potential can be created by focusing blue-detuned far-off-resonant laser light into the center of the magnetic trap [13], or by deforming single-well optical trap into a double-well potential with linearly increasing the frequency difference between the rf-signals [14]. In both systems the interference between the two condensates was observed by simultaneously switching off the magnetic or optical trap and the laser-light sheet. To observe the fidelity instability due to the internal dynamics, before switching off the traps, we apply near-resonant coupling fields to condensates to couple the two hyperfine state of ^{87}Rb , e.g., $F = 1$, $m_F = -1$ and $F = 2$, $m_F = +1$ like [12], with a slight different strength in the two wells. The condensate in each well can then dynamically convert between internal states. Then switching off the traps and letting the two clouds of BEC expand, our prediction of the fast decay of system’s fidelity will lead to the fadeaway of the inference pattern.¹ Here we require the wells be deep, so that the total density remains relatively constant. The atom number in each well is nearly equal. Small fluctuation on the atom number does not essentially affect our predictions since it may be regarded as another perturbation source.

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¹ This can be understood from the formula, $I \propto |\psi_1 \chi_1|^2 + |\psi_2 \chi_2|^2 + 2 \text{Re}(\psi_1^* \psi_2 \chi_1^* \chi_2)$ with 1, 2 label the wells. Clearly, high fidelity of the two internal states χ corresponds to high visibility of the interference.

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