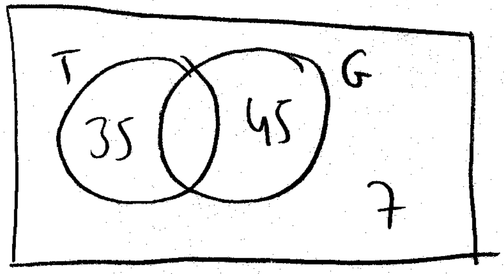
List of formulas

$I = P.r.t \qquad A = P.(1 + r.t) \qquad A = P(1 + \frac{r}{m})^{m.t} \qquad APY = (1 + \frac{r}{m})^m - 1$

$FV = PMT\,\frac{(1+i)^n - 1}{i} \qquad PV = PMT\,\frac{1-(1+i)^{-n}}{i} \qquad i = \frac{r}{m} \qquad n = m.t$

Problem 1 a) (7 pts) A group of 100 people includes 35 who play only tennis, 45 who play only golf and 7 who play neither sport. How many people in the group play both tennis and golf?



$35+45+7=87$

$100-87=\boxed{13}$

b) (8 pts) From a committee of 10 people, in how many ways can we choose a chairperson (baskan), a vice-chairperson (baskan yardimcisi) and 3 members?

$\binom{10}{1}\cdot\binom{9}{1}\cdot\binom{8}{3} = 10\cdot 9\cdot \frac{8\cdot 7\cdot 6\cdot \cancel{5!}}{3!\,\cancel{5!}} = 10\cdot 9\cdot 8\cdot 7$

$= 5040$

Problem 2 a) (10 pts) A family wants to buy a house whose price is \$100,000. How much downpayment (pesinat) should they pay to buy the house with \$600 monthly payment for 20 year mortgage at 6% compounded monthly.

$$PV = PMT \frac{1-(1+i)^{-n}}{i}$$

$$= 600 \, \frac{1-(1{,}005)^{-240}}{0{,}005}$$

$$= 83\,748{,}46$$

$$\text{Downpayment} = 100\,000 - 83\,748{,}46$$

$$= \boxed{16\,251{,}53}$$

b) (8 pts) In the above question, calculate how much interest is paid at the end of 10 years.

0	10	20
83 748,46	X	0

$$X = 600 \, \frac{1-(1{,}005)^{-120}}{0{,}005}$$

$$= 54\,044{,}07$$

$$\begin{array}{r} 83\,748{,}46 \\ -\ 54\,044{,}07 \\ \hline 29\,704{,}39 \end{array} \rightarrow$$ money gone to house

$120 \times 600 = 72\,000 \rightarrow$ money paid

$$72\,000 - 29\,704{,}39 = \boxed{42\,295{,}61}$$

Problem 3 (15 pts) Solve the following system by using Gauss-Jordan elimination.

$$\begin{aligned} 3x - 2y + \ z &= -7 \\ 2x + \ y - 4z &= \ 0 \\ x + \ y - 3z &= \ 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -7 \\ 2 & 1 & -4 & 0 \\ 1 & 1 & -3 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 2 & 1 & -4 & 0 \\ 3 & -2 & 1 & -7 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & -1 & 2 & -2 \\ 0 & -5 & 10 & -10 \end{array}\right] -R_2 \to R_2 \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -5 & 10 & -10 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 5R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \infty \text{ solutions}$$

$$\begin{aligned} x - z &= -1 \\ y - 2z &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= z - 1 \\ y &= 2z + 2 \end{aligned}$$

$$(z-1, 2z+2, z)$$

List of formulas

$(x^n)' = n.x^{n-1}$ $\qquad (e^x)' = e^x$ $\qquad (\ln x)' = \frac{1}{x}$

$(f(x).g(x))' = f'(x).g(x) + f(x).g'(x)$ $\qquad \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x).g(x) - f(x).g'(x)}{g^2(x)}$

Problem 4 (10 pts) Find two negative numbers whose product is 30 and their sum is maximum.

$x \cdot y = 30$ $\qquad$ Maximize: $x+y$

$y = \frac{30}{x}$ $\qquad f(x) = x + \frac{30}{x}$

$f'(x) = 1 - \frac{30}{x^2} \Rightarrow f'(x) = 0 \Leftrightarrow \frac{x^2 - 30}{x^2} = 0$

$x^2 = 30$

$x = \sqrt{30}$ $\boxed{x = -\sqrt{30}}$

x		$-\sqrt{30}$		0		$\sqrt{30}$	
f'	+		−		−		+
f	↗		↘		↘		↗

x,y negative:

$x = -\sqrt{30}$

$y = -\sqrt{30}$

Problem 5 a) (8 pts) If f is continuous everywhere, find a and b.

$$f(x) = \begin{cases} x^2 - ax + b & x > 2 \\ bx - 3a & -1 \le x \le 2 \\ 8x - a + b & x < -1 \end{cases}$$

$x=2$ $\quad 4-2a+b = 2b-3a \Rightarrow a-b=-4$

$x=-1$ $\quad -b-3a = -8-a+b \Rightarrow a+b=4$

$\Downarrow$

$\boxed{a=0 \quad b=4}$

b) (7 pts) In the question above, is f differentiable everywhere?

Points to check: $x=2$ $\quad x=-1$

$x=2$ $\quad$ Right $f'(x)=2x \Rightarrow f'(2)^+=4$ ✓

Left $f'(x)=4 \Rightarrow f'(2)^-=4$

$x=-1$ $\quad$ Right $f'(x)=4 \Rightarrow f'(-1)^+=4$ $\neq$ corner!

Left $f'(x)=8 \Rightarrow f'(-1)^-=8$

f is not differentiable at -1.

Problem 6 (15 pts) Let $f(x) = \frac{x-2}{x^2}$

a) Find all local extrema and intervals on which f is increasing & decreasing.

$$f'(x) = \frac{1\cdot x^2 - (x-2)\cdot 2x}{x^4} = \frac{x^2 - 2x^2 + 4x}{x^4} = \frac{-x^2+4x}{x^4} = \frac{-x+4}{x^3}$$

f': $-$ on $(-\infty, 0)$ ↘, $+$ on $(0, 4)$ ↗, $-$ on $(4, \infty)$ ↘

$\boxed{4} \rightarrow$ local max

0 not defined! $(4, \frac{1}{8})$

b) Find inflection points, and intervals on which f is concave up & concave down.

$$f''(x) = \frac{-1\cdot x^3 - (-x+4)\cdot 3x^2}{x^6} = \frac{-x^3 + 3x^3 - 12x^2}{x^6} = \frac{2x^3 - 12x^2}{x^6} = \frac{2(x-6)}{x^4}$$

f'': $-$ concave down, 0, $-$ con. down, 6, $+$ con. up

$\boxed{6} \rightarrow$ inflection pt.

$(6, \frac{1}{9})$

c) Find the asymptotes, if exist.

$$\lim_{x\to\infty} \frac{x-2}{x^2} = 0$$

$$\lim_{x\to-\infty} \frac{x-2}{x^2} = 0$$

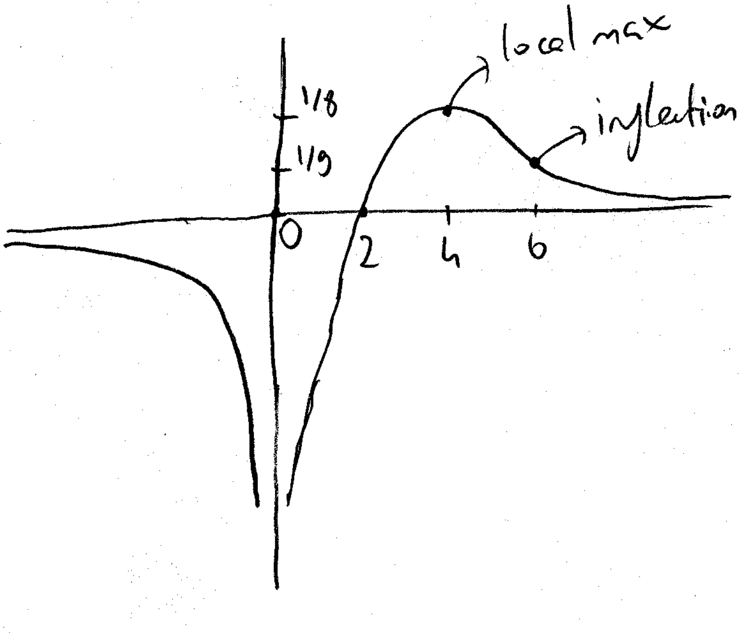
$\Rightarrow y = 0$ hor. asymptote

$x = 0$ vertical asymptote

$$\lim_{x\to 0^+} \frac{x-2}{x^2} = -\infty$$

$$\lim_{x\to 0^-} \frac{x-2}{x^2} = -\infty$$

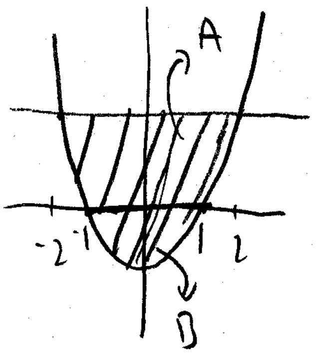
e) Sketch the graph of f.



List of formulas

$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int e^x dx = e^x + C \qquad \int \frac{1}{x} dx = \ln x + C$

Problem 7 a) (10 pts) Find the area of the region between $y = x^2 - 1$ and $y = 3$.



$-B = \int_{-1}^{1} x^2 - 1\,dx = \frac{x^3}{3} - x \Big|_{-1}^{1} = \left(\frac{1}{3} - 1\right) - \left(\frac{-1}{3} + 1\right) = -4/3$

$\Rightarrow B = 4/3$

$A = 12 - \left(\int_{1}^{2} x^2 - 1\,dx + \int_{-2}^{-1} x^2 - 1\,dx\right) = 12 - \frac{8}{3} = \frac{28}{3}$

Area = A+B

$\text{Area} = \frac{4}{3} + \frac{28}{3} = \frac{32}{3}$

b) (7 pts) $\int_2^5 \frac{x^2 - 3}{\sqrt{x}}\, dx$

$\int_2^5 x^{3/2} - 3x^{-1/2}\,dx = \frac{x^{5/2}}{5/2} - 3\frac{x^{1/2}}{1/2} \Big|_2^5 = \frac{2}{5} x^{5/2} - 6x^{1/2} =$

$= \left(\frac{2}{5} 5^{5/2} - 6 \cdot 5^{1/2}\right) - \left(\frac{2}{5} 2^{5/2} - 6 \cdot 2^{1/2}\right)$

$= \left(10\sqrt{5} - 6\sqrt{5}\right) - \left(\frac{8}{5}\sqrt{2} - 6\sqrt{2}\right)$

$= 4\sqrt{5} + \frac{22}{5}\sqrt{2}$