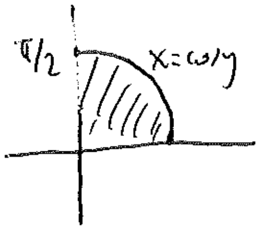
Problem 1) (20 pts) Evaluate the following double integrals.

a.) $\iint\limits_D x \sin y \, dA$ where $D = \{(x,y) \mid 0 \le y \le \pi/2,\ 0 \le x \le \cos y\}$



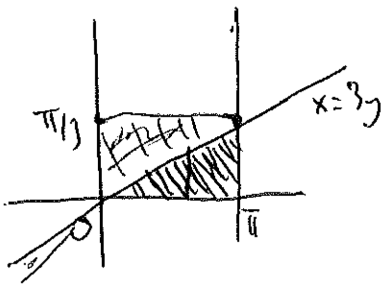
$$I = \int_0^{\pi/2}\int_0^{\cos y} x \sin y \, dx\, dy = \int_0^{\pi/2} \left[\frac{x^2}{2} \sin y \Big|_0^{\cos y} \right] dy$$

$$= \int_0^{\pi/2} \frac{\cos^2 y}{2} \sin y \, dy = -\frac{1}{2} \int_1^0 u^2 \, du = -\frac{u^3}{6} \Big|_1^0 = 1/6$$

$u = \cos y$

$du = -\sin y \, dy$

b.) $\int_0^{\pi/3} \int_{3y}^{\pi} \cos(x^2) \, dx dy$ *Hint: Change the order of integration.*



$$I = \int_0^{\pi}\int_0^{x/3} \cos x^2 \, dy\, dx = \int_0^{\pi} \left[y \cos x^2 \Big|_0^{x/3} \right] dx$$

$$= \int_0^{\pi} \frac{x \cos x^2}{3} \, dx = \int_0^{\pi^2} \frac{\cos u}{6} \, du = \frac{\sin u}{6} \Big|_0^{\pi^2}$$

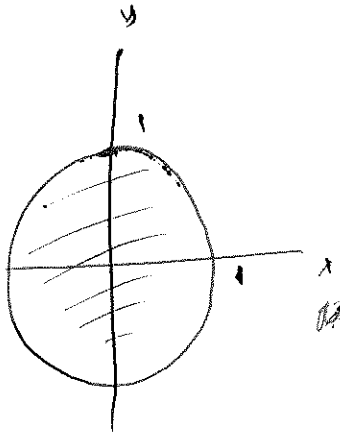
$u = x^2$

$du = 2x \, dx$

$$= \frac{\sin \pi^2}{6}$$

Problem 2. (15 pts) Evaluate the following double integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{x^2+y^2+1}\,dx\,dy$$



polar coordinates:

$$\int_0^{2\pi}\int_0^1 \frac{1}{r^2+1}\, r\,dr\,d\theta$$

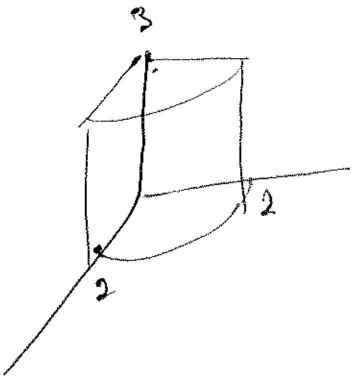
$r^2+1=u$

$2r\,dr = du$

$$= \int_0^{2\pi}\left(\frac{\ln(r^2+1)}{2}\Big|_0^1 \right) d\theta$$

$$= 2\pi \frac{\ln 2}{2} = \pi \ln 2.$$

Problem 3.) (18 pts) Write 6 different (order) iterated triple integrals in cartesian coordinates for the volume of the solid in the first octant $\{x \geq 0,\ y \geq 0,\ z \geq 0\}$, enclosed by the cylinder $x^2 + y^2 = 4$, and the plane $z = 3$.



① $$\int_0^3 \int_0^2 \int_0^{\sqrt{4-y^2}} dx\,dy\,dz$$

② $$\int_0^3 \int_0^2 \int_0^{\sqrt{4-x^2}} dy\,dx\,dz$$

③ $$\int_0^2 \int_0^3 \int_0^{\sqrt{4-x^2}} dy\,dz\,dx$$

④ $$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^3 dz\,dx\,dy$$

⑤ $$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^3 dz\,dy\,dx$$

⑥ $$\int_0^2 \int_0^3 \int_0^{\sqrt{4-y^2}} dx\,dz\,dy$$

Problem 4) (15 pts) Let E be the solid lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above the xy-plane, and below the plane $z = x + 2$. Evaluate the following triple integral.

$$\iiint_E x \, dV$$

$$= \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta + 2} r\cos\theta \; r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(r^2 \cos\theta \, z \Big|_{z=0}^{z=r\cos\theta+2} \right) dr \, d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^3\cos^2\theta + 2r^2\cos\theta \; dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^4\cos^2\theta}{4} + \frac{2r^3\cos\theta}{3} \Big|_{r=1}^{r=2} \right) d\theta$$

$$= \int_0^{2\pi} 4\cos^2\theta + \frac{16\cos\theta}{3} - \frac{\cos^2\theta}{4} - \frac{2\cos\theta}{3} \; d\theta$$

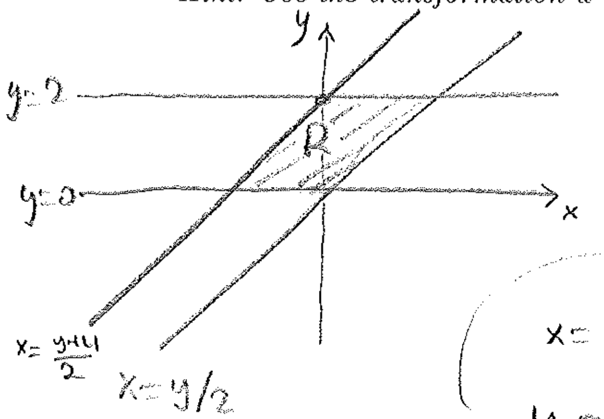
$$= \int_0^{2\pi} \left(\frac{15}{4} \cdot \left(\frac{\cos 2\theta + 1}{2} \right) + \frac{14}{3} \cos\theta \right) d\theta$$

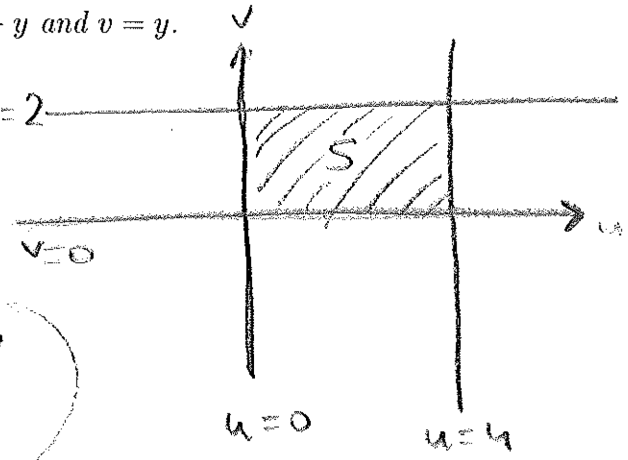
$$= \frac{15 \sin 2\theta}{16} + \frac{15\theta}{8} + \frac{14}{3} \sin\theta \Big|_0^{2\pi} = \frac{15\pi}{4}$$

Problem 5) (15 points) Evaluate the following double integral.

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dxdy \quad =$$

Hint: Use the transformation $u = 2x - y$ and $v = y$.





$$x = \frac{u+v}{2}$$
$$y = v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$= \int_0^2 \int_0^4 v^3\, u\, e^{u^2}\, \frac{1}{2}\, du\, dv$$

$$= \int_0^2 \left(\frac{v^3 e^{u^2}}{4} \Bigg|_{u=0}^{u=4} \right) dv = \int_0^2 \left(\frac{e^{16}-1}{4} \right) v^3\, dv$$

$$= \frac{e^{16}-1}{16} v^4 \Bigg|_0^2 = e^{16}-1$$

Problem 6) (10 pts) Evaluate the following line integral where C is the straight line segment from $(1, 0, 1)$ to $(0, 3, 6)$.

$$\int_C xy^2 z \, ds$$

A parametrization of C is $r(t) = \langle 1-t, 3t, 1+5t \rangle$, $0 \le t \le 1$

Therefore

$$\int_C xy^2 z \, ds = \int_0^1 (1-t)(3t)^2(1+5t)\sqrt{(-1)^2+(3)^2+(5)^2}\, dt$$

$$= \int_0^1 9\sqrt{35}\left(-5t^4+4t^3+t^2\right) dt$$

$$= 9\sqrt{35}\left(-t^5+t^4+\frac{t^3}{3}\right)\Big|_0^1$$

$$= 9\sqrt{35} \cdot \frac{1}{3} = 3\sqrt{35}$$

$$2e^{2y} \overset{\checkmark}{=} e^{2y}$$

$\partial/\partial y \qquad \partial/\partial x$

Problem 7) (12 pts) Let $\mathbf{F}(x,y) = < e^{2y}, 1 + 2xe^{2y} >$ be a vector field in $\mathbf{R}^2$.

Let $\mathbf{r}(t) = < te^t, 1+t >, \quad 0 \le t \le 1$ be a parametrization of the curve C.

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\boxed{f(x,y) = xe^{2y} + g(y)} \longrightarrow f_y = 2xe^{2y} + g'(y) = 1 + 2xe^{2y}$$

$$g'(y) = 1 \Rightarrow \boxed{g(y) = y + \text{const.}}$$

For $f(x,y) = xe^{2y} + y$, $\nabla f = F$

Hence, by the Fund. Thm for line int.

$$\int_C F \cdot dr = f(r(1)) - f(r(0))$$

$$= f(e,2) - f(0,1)$$

$$= e^5 + 1$$

OR:

$$\int_C F \cdot dr = \int_0^1 \langle e^{2+2t}, 1 + 2te^t e^{2+2t} \rangle \cdot \langle te^t + e^t, 1 \rangle \, dt$$

$$= \int_0^1 \left(te^{2+3t} + e^{2+3t} + 1 + 2te^{2+3t} \right) dt$$

$$= \int_0^1 \left((1+3t)e^{2+3t} + 1 \right) dt$$

$$= te^{2+3t} + t \Big|_0^1 = e^5 + 1$$