Problem 1)(12 pts) Find the absolute maximum and minimum values of

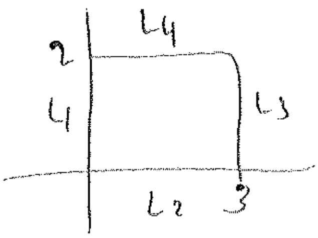
$$f(x,y) = x^4 + y^4 - 4xy + 1$$

on the closed region $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Critical Points :

$$\left.\begin{aligned} f_x &= 4x^3 - 4y = 0 \\ f_y &= 4y^3 - 4x = 0 \end{aligned}\right\} \quad \left.\begin{aligned} y &= x^3 \\ y^3 &= x \end{aligned}\right\} \to (x,y) = \boxed{(0,0)},\ \boxed{(1,1)},\ = (-1,-1) \to \notin D.$$

On the boundary of D



on L_1 : $f(0,y) = y^4 - 1$, $f' = 4y^3 = 0 \to y = 0$ $\Big\}$ $\boxed{(0,0)}$, $\boxed{(0,2)}$ endpoint

on L_2 : $f(x,0) = x^4 - 1$, $f' = 4x^3 = 0 \to x = 0$ $\Big\}$ $\boxed{(0,0)}$, $\boxed{(3,0)}$

on L_3 : $f(3,y) = 82 + y^4 - 12y$, $f' = 4y^3 - 12 = 0 \to y = \sqrt[3]{3}$ $\Big\}$ $\boxed{(3, \sqrt[3]{3})}$, $(3,0)$, $\boxed{(3,2)}$

on L_4 : $f(x,2) = x^4 + 17 - 8x$, $f' = 4x^3 - 8 = 0 \to x = \sqrt[3]{2}$ $\Big\}$ $\boxed{(\sqrt[3]{2}, 2)}$, $\boxed{(0,2)}$, $(3,2)$

$f(0,0) = 1$

$\boxed{f(1,1) = -1}$ → abs. min.

$f(0,2) = 17$

$\boxed{f(3,0) = 82.}$ → abs. max.

$f(3, \sqrt[3]{3}) = 82 - 9\sqrt[3]{3}$

$f(\sqrt[3]{2}, 2) = 17 - 6\sqrt[3]{2}$

$f(3,2) = 74$

Problem 2a.)(7 pts) At what point on the paraboloid $y = x^2 + z^2 + 2$ is the tangent plane parallel to the plane $4x + y + 6z = 12$?

$y - x^2 - z^2 = 2$

Note that $\nabla f\big|_{(x_0,y_0,z_0)}$ is normal to the tangent plane of $f(x,y,z)$ at the point (x_0,y_0,z_0).

$$\nabla f\big|_{(x_0,y_0,z_0)} = (-2x_0, 1, -2z_0)$$

Then if $(4k, k, 6k) = (-2x_0, 1, -2z_0)$ we have $k=1$, $x_0 = -2$, $z_0 = -3$

Since (x_0, y_0, z_0) is on the paraboloid $y = x^2 + z^2 + 2$, we must have $y_0 = (-2)^2 + (-3)^2 + 2 = 15$, so the point is $(-2, 15, -3)$.

2b.)(7 pts) Find $f_x(1,0)$ if $f(x,y) = x(x^2+y^2)^{-3/2}e^{\sin(x^2y)}$.

Hint: You may prefer to use the definition of the partial derivative.

First Method

$$\lim_{h\to0}\frac{f(x+h,y)-f(x,y)}{h} =$$

assuming the partial derivative exists

$$\Rightarrow \lim_{h\to0}\frac{f(x+h,0)-f(x,0)}{h} = f_x(1,0)$$

$$= \lim_{h\to0}\frac{(x+h)(x+h)^{-3} - x\cdot x^{-3}}{h}$$

$$= \lim_{h\to0}\frac{(x+h)^{-2} - x^{-2}}{h} \Rightarrow (x^{-2})'\Big|_{x=1} = (-2x^{-3})\Big|_{x=1}$$

$$= -2$$

Second Method

$$f_x(x,y) = \left(x(x^2+y^2)^{-3/2}\right)' e^{\sin(x^2y)} + x(x^2+y^2)^{-3/2}\cdot\left(e^{\sin(x^2y)}\right)'$$

$$\Rightarrow f_x(1,0) = \left(\left(x(x^2+y^2)^{-3/2}\right)' + \left(e^{\sin(x^2y)}\right)'\right)\Big|_{\substack{x=1\\y=0}}$$

$$= \left((x^2+y^2)^{-3/2} + x\left((x^2+y^2)^{-3/2}\right)' + \underbrace{\cancel{\cos(x^2y)\cdot 2xy\cdot e^{\sin(x^2y)}}}_{0}\right)\Big|_{\substack{x=1\\y=0}}$$

$$= \left(1 + \left(\frac{-3}{2}\right)(x^2+y^2)^{-5/2}\cdot 2\cdot x\right)\Big|_{\substack{x=1\\y=0}}$$

$$= 1 - \frac{3}{2}\cdot 2 = -2$$

Problem 3a.)(6 pts) Consider the region R whose area is given by the integral

$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$$

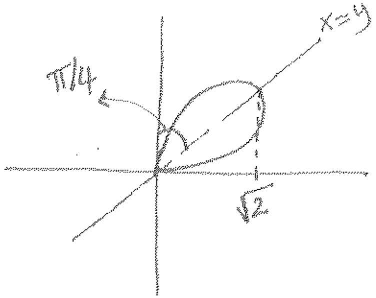
comes from changing into polar coordinates

Which of the points with cartesian coordinates $\left(-\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$, $\left(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$ and $\left(-\frac{1}{2\sqrt{2}},0\right)$ belong to R?

2 points 2 points 2 points

$0 \le \theta \le \pi/2$ → This indicates that the region is in the first quadrant. So the points $\left(\frac{-1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$ & $\left(\frac{-1}{\sqrt{2}},0\right)$ cannot belong to the region R.

$0 \le r \le \sin 2\theta$



For the point $\left(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$, consider $r=\sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\sqrt{2}}\right)^2}=\frac{1}{2}$

& $\tan\theta=\frac{y}{x}=\frac{1/2\sqrt{2}}{1/2\sqrt{2}}=1$ which implies $\theta=45^\circ=\pi/4$

so that $\left(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$ belongs to R.

3b.)(6 pts) Consider the solid E whose volume is given by the integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

comes from coordinate change

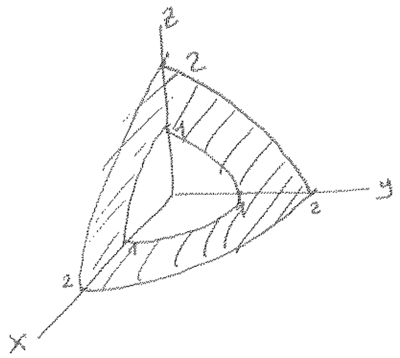
2 pt 2 pt 2 pt

Which of the points with cartesian coordinates $(1,1,2)$, $\left(\frac{3}{4},\frac{3}{4},\frac{3\sqrt{2}}{4}\right)$ and $(1,1,1)$ belong to E?

$1 \le \rho \le 2$ → indicates that the region is between the spheres $\rho=1$ & $\rho=2$

$0 \le \phi \le \pi/2$ → indicates that the region is on the upper half space

$0 \le \theta \le \pi/2$ → indicates that the region is in the first octant



every point (given) is in the first octant. So it is enough to find their distances to zero (the value ρ).

[For $(1,1,2)$] $\Rightarrow$ $\rho=\sqrt{1^2+1^2+2^2}=\sqrt{6}\notin[1,2]$

[For $\left(\frac{3}{4},\frac{3}{4},\frac{3\sqrt{2}}{4}\right)$] $\Rightarrow$ $\rho=\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^2+\left(\frac{3\sqrt{2}}{4}\right)^2}=\sqrt{\frac{36}{16}}=\frac{6}{4}\in[1,2]$

[For $(1,1,1)$] $\Rightarrow$ $\rho=\sqrt{1^2+1^2+1^2}=\sqrt{3}\in[1,2]$

Therefore $(1,1,2)$ doesnot belong to the region and the points $(1,1,1)$ & $(3/4, 3/4, 3\sqrt{2}/4)$ belong to the region E.

Problem 4) (10 pts) Find the volume of the region between the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

$z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2 \Rightarrow x^2 + y^2 = 36 - 3x^2 - 3y^2$

$\Rightarrow x^2 + y^2 = 9$

$$\Rightarrow V = \iint_D \left[(36 - 3x^2 - 3y^2) - (x^2 + y^2)\right] dA$$

$D = \{(x, y) : x^2 + y^2 \leq 9\}$

$x = r\cos\theta$
$y = r\sin\theta$

$D = \{(r, \theta) : 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$

$$\Rightarrow V = \int_0^{2\pi}\int_0^3 (36 - 4r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[18r^2 - r^4\right]_0^3 d\theta$$

$$= \int_0^{2\pi} 81\, d\theta$$

$$= 81\,\theta \Big|_0^{2\pi}$$

$$= 162\pi$$

Problem 5) (10 points) For each of the following vector fields, determine whether or not it is conservative, and find a function f such that $\mathbf{F} = \nabla f$ if $\mathbf{F}$ is conservative.

a.) $\mathbf{F}(x, y, z) = \langle ye^{-x}, e^{-x}, 2z \rangle$

If F is conservative then curl F = 0

$P = ye^{-x}$ $\quad Q = e^{-x}$ $\quad R = 2z$

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k$$

$$= 0i + 0j - 2e^{-x}k$$

$$\neq 0$$

So, F is not conservative

b.) $\mathbf{F}(x, y, z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

If curl F = 0 and the component functions of F have continuous partial derivatives then F is conservative.

$$\text{curl } F = \left(xe^{yz} + xyze^{yz} - xe^{yz} - xyze^{yz}\right)i + \left(ye^{yz} - ye^{yz}\right)j + \left(ze^{yz} - ze^{yz}\right)k$$

$$= 0$$

So, F is conservative. There exists f such that $\nabla f = F$

$\Rightarrow f_x = e^{yz} \quad \Rightarrow f = xe^{yz} + g(y,z)$

$f_y = xze^{yz} \quad \Rightarrow f_y = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$f_z = xye^{yz} \quad \Rightarrow f_z = xye^{yz} + h'(z) = xye^{yz} \Rightarrow h'(z) = 0 \Rightarrow h(z) = c$ constant

$\Rightarrow f(x,y,z) = xe^{yz} + c$

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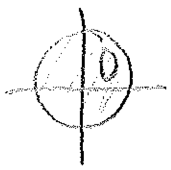
Problem 6a.) (8 pts) Find the area of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.



$f(x,y) = xy \Rightarrow$ Area: $\iint_D dS = \iint_D \sqrt{1+f_x^2+f_y^2}\,dx\,dy$ 3

$= \iint_D \sqrt{1+x^2+y^2}\,dx\,dy$ 1

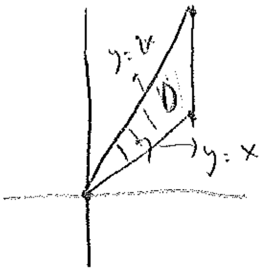
$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{(1+r^2)^{3/2}}{3/2\cdot 2}\Big|_0^1\right]d\theta$

$= \int_0^{2\pi} \frac{2\sqrt{2}-1}{3}\,d\theta = \frac{2\sqrt{2}-1}{3}\cdot 2\pi$ // 3

6b.) (8 pts) Evaluate the following integral.

$$\int_C xy^2\,dx + 2x^2y\,dy\ ,$$

where C is the positively oriented triangle with vertices $(0,0), (2,2)$ and $(2,4)$.



Green's theorem :)

$I = \iint_D (4xy - 2xy)\,dx\,dy = \int_0^2\int_x^{2x} 2xy\,dy\,dx$ 5

$= \int_0^2 \left[xy^2\Big|_x^{2x}\right]dx = \int_0^2 (4x^3 - x^3)\,dx = \int_0^2 3x^3\,dx = \frac{3x^4}{4}\Big|_0^2$

$= 12 - 0 = 12$ // 3

Problem 7a.) (4 pts) Verify that curlF $= \langle -x, -y, 2z\rangle$ for $\mathbf{F}(x, y, z) = \langle -yz, xz, z^3\rangle$.

$$\nabla \times \vec{F} : \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^3 \end{vmatrix} = \langle 0 - x, -y - 0, z + z\rangle = \langle -x, -y, 2z\rangle$$

7b.) (12 pts) Let C be the positively oriented curve which consists of two smooth pieces: the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$ followed by the line segment from $(1, 0, 2\pi)$ to $(1, 0, 0)$. Provided that $\mathbf{G}(x, y, z) = \langle -x, -y, 2z\rangle$ and S is an oriented piecewise-smooth surface bounded by C, find $\iint_S \mathbf{G} \cdot d\mathbf{S}$.

$$\iint_S G \cdot dS = \iint_S \text{curl}\vec{F} \cdot dS \overset{\text{Stoke's}}{=} \oint_C \vec{F} \cdot d\vec{r}$$ by Stoke's Theorem where $C = \partial S$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$C_1 : \vec{r}_1(t) = \langle \cos t, \sin t, t\rangle \quad 0 \le t \le 2\pi$

$C_2 = \vec{r}_2(t) = \langle 1, 0, 2\pi - t\rangle \quad 0 \le t \le 2\pi$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \langle -yz, xz, z^3\rangle \cdot \vec{r}_1'(t)\, dt = \int_0^{2\pi} \langle -\sin t \cdot t, \cos t \cdot t, t^3\rangle \cdot \langle -\sin t, \cos t, 1\rangle\, dt$$

$$= \int_0^{2\pi} \sin^2 t \cdot t + \cos^2 t \cdot t + t^3\, dt = \int_0^{2\pi} t + t^3\, dt = \frac{t^2}{2} + \frac{t^4}{4} \Big|_0^{2\pi} = 2\pi^2 + 4\pi^4$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -yz, xz, z^3\rangle \cdot \underbrace{\langle 0, 0, -1\rangle}_{\vec{r}_2'(t)}\, dt = -\int_0^{2\pi} (2\pi - t)^3\, dt = \int_{2\pi}^{0} u^3\, du = \frac{u^4}{4}\Big|_{2\pi}^{0} = -4\pi^4$$

$u = 2\pi - t$

$\Rightarrow \quad I = 2\pi^2 + 4\pi^4 - 4\pi^4$ 8

$= 2\pi^2$ //

Problem 8) Let $\mathbf{F}(x,y,z) = \langle xz^2, yx^2, zy^2 \rangle$, and S be the sphere $x^2+y^2+z^2=4$ with its positive orientation.

8a.) (10 pts) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$$\iint_S F \cdot dS = \iiint_R \operatorname{div} F \, dV$$ by Divergence theorem. R is the ball of radius 2.

3

$$\operatorname{div} F = z^2 + x^2 + y^2 \Rightarrow \iiint_R (x^2+y^2+z^2)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$ spherical coordinates

4

$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi \left[\frac{\rho^5}{5}\Big|_0^2 \right] d\phi\, d\theta = \frac{32}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi\, d\theta = \frac{32}{5}\int_0^{2\pi} \left[-\cos\phi \Big|_0^{\pi} \right] d\theta$$

$$= \int_0^{2\pi} \frac{64}{5}\, d\theta = \frac{128\pi}{5}$$ //

8b.) (5 pts) Find $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.

$$\iint_S \operatorname{curl} F \cdot dS = \iiint_R \operatorname{div}(\operatorname{curl} F)\, dV$$

$\operatorname{div}(\operatorname{curl} F) = 0$ for any F $\Rightarrow$ $I = 0$