Problem 1. a) (10 pts) Find the general solution of the following equation.

(Hint: Try to find an integrating factor which depends only on y.)

$$\underbrace{\left(\frac{4x^3}{y^2}+\frac{3}{y}\right)}_{M} dx+\underbrace{\left(\frac{3x}{y^2}+4y\right)}_{N} dy=0$$

$M_y = \dfrac{-8x^3}{y^3} - \dfrac{3}{y^2}$ $\quad N_x = \dfrac{3}{y^2}$ $\quad M_y - N_x = -\dfrac{8x^3}{y^3} - \dfrac{6}{y^2} \Rightarrow \dfrac{N_x - M_y}{M} = \dfrac{2}{y}$

$\dfrac{\mu'(y)}{\mu} = \dfrac{2}{y} \Rightarrow \mu(y) = y^2 \Rightarrow \underbrace{(4x^3+3y)}_{\hat{M}}dx + \underbrace{(3x+4y^3)}_{\hat{N}}dy = 0$ exact

$\hat{M}_y = 3 \qquad \hat{N}_x = 3$ ✓

$\psi_x = 4x^3+3y \Rightarrow \psi = x^4+3xy+f(y)$

$\psi_y = 0+3x+f'(y) = 3x+4y^3 \Rightarrow f(y) = y^4$

$\psi = x^4+3xy+y^4$

Solution $\quad x^4+3xy+y^4 = C$

(i) $y(1)=1 \Rightarrow 1+3+1=5 \Rightarrow C=1$ $\qquad 4x^3+3y+3xy'+4y^3y'=0$

$y' = -\dfrac{4x^3+3y}{3x+4y^3} = -\dfrac{7}{7} = -1 < 0$ ✓

(ii) $C=5 \Rightarrow 0+0+y^4=5 \Rightarrow y=\pm\sqrt[4]{5} < 2$

b) (*no explanation required, 2 points each*)

True or false:

(i) If y is a solution of the equation above with $y(1)=1$, then $y'(1)>0$. **T** **F**

(ii) If y is a solution of the equation above with $y(1)=1$, then $y(0)<2$. **T** **F**

10 8

Problem 2. a) ($\cancel{10}$ pts) Find the general solution of the following equation.

$$y''' - 3y'' + 2y' = t + e^t$$

$$D(D^2 - 3D + 2)y = D(D-2)(D-1)y = t + e^t \Rightarrow y_H = c_1 + c_2 e^t + c_3 e^{2t}$$

$$\Rightarrow \underbrace{D^2 \cdot (D-1)}_{\text{annihilator}} \cdot D(D-1)(D-2)y = 0 \Rightarrow D^3(D-1)^2(D-2)y = 0$$

$$\Rightarrow y_H = c_1 + \underline{c_2 t} + \underline{c_3 t^2} + c_4 e^t + \underline{c_5 te^t} + c_6 e^{2t}$$

$y_p = ?$ $\quad y_{P_1} = c_2 t + c_3 t^2 \Rightarrow 0 - 3(2c_3) + 2(c_2 + 2c_3 t) = t$

$$\Rightarrow c_3 = \frac{1}{4} \quad c_2 = \frac{3}{4}$$

$$y_{P_1} = c_5 te^t \Rightarrow c_5\{(t+3)e^t - 3(t+2)e^t + 2(t+1)e^t\} = e^t \Rightarrow c_5 = -1$$

$$y_p = \frac{t^2}{4} + \frac{3t}{4} - te^t$$

$$y = c_1 + c_2 e^t + c_3 e^{2t} + \frac{t^2}{4} + \frac{3t}{4} - te^t$$

b) (*no explanation required, 2 points each*)

True or false:

(i) $y = \frac{3t}{4} + e^t - e^{2t}$ is a solution of the equation above. T F

(ii) $y = \frac{3t}{4} + \frac{t^2}{4} - te^t + 2 - e^t$ is a solution of the equation above. T F

Problem 3. a) (~~10~~ 8 pts) Find the general solution of the following equation.

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

VARIATION of PARAMETERS: $y_H = c_1 \underbrace{e^t}_{y_1} + c_2 \underbrace{te^t}_{y_2}$

$y = u_1 e^t + u_2 te^t$

$u_1 = -\int \frac{y_2 \cdot g}{W} \quad u_2 = \int \frac{y_1 \cdot g}{W} \qquad W = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}$

$\Rightarrow \quad u_1 = -\int \frac{te^t \cdot e^t/1+t^2}{e^{2t}} = -\int \frac{t}{1+t^2} = -\frac{\ln(1+t^2)}{2}$

$u_2 = \int \frac{e^t \cdot e^t/1+t^2}{e^{2t}} = \int \frac{1}{1+t^2} = \arctan t$

$\Rightarrow \quad y = \left(-\frac{\ln(1+t^2)}{2} \cdot e^t\right) + \left(\arctan t \cdot te^t\right) + c_1 e^t + c_2 te^t$

b) (*no explanation required, 2 points each*)

True or false:

(i) $\{2e^t, e^t + te^t\}$ is a fundamental set of solutions of $y'' - 2y' + y = 0$ T F

(ii) $y = \frac{3t}{4} + \frac{t^2}{4} - te^t + 2 - e^t$ is a solution of $y'' - 2y' + y = \frac{e^t}{1+t^2}$. T F

Problem 4. a) ($\cancel{10}$ 8 pts) Determine the first 4 terms of the power series solution of the following initial value problem.

$$(2+x^2)y'' - xy' + 4y = 0 \ , \qquad y(0) = 2 \ , \ \ y'(0) = 4$$

$\phi = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \Rightarrow$ $a_0 = \phi(0) = 2$, $a_1 = \phi'(0) = 4$, $a_2 = \frac{\phi''(0)}{2}$, $a_3 = \frac{\phi'''(0)}{6}$

$(2+x^2)\phi'' - x\phi' + 4\phi = 0$

$x=0 \Rightarrow$ $2\phi'' - 0 + 4\phi = 0 \Rightarrow \phi''(0) = -4\phi(0) = -4.2 = -8$

$2x\phi'' + (2+x^2)\phi''' - [\phi' + x\phi''] + 4\phi' = 0$

$(2+x^2)\phi''' + x\phi'' + 3\phi' = 0$

$x=0$ $\quad 2\phi'''(0) + 0 + 3\phi'(0) = 0 \Rightarrow \phi'''(0) = \frac{-3}{2}\phi'(0) = \frac{-3}{2}.4 = -6$

$a_2 = \frac{\phi''(0)}{2} = \frac{-8}{2} = -4$

$a_3 = \frac{\phi'''(0)}{6} = \frac{-6}{6} = -1$

$\Rightarrow \phi = 2 + 4x - 4x^2 - 6x^3$

b) (*no explanation required, 2 points*) True or false:

The power series solution for the above initial value problem is convergent at $x = 1$. **T** F

$1 < \sqrt{2}$

Problem 5. a) (~~8~~ 6 pts) Determine the function whose Laplace transform is $\frac{e^{-2s}(2s-4)}{s^2-4s+3}$
(Hint: You can use the table on the last page of the exam.)

$$\frac{e^{-2s}(2s-4)}{s^2-4s+3} = e^{-2s}\left[\frac{A}{s-3}+\frac{B}{s-1}\right] = e^{-2s}\left[\frac{1}{s-3}+\frac{1}{s-1}\right]$$

$$\left[\frac{1}{s-3}+\frac{1}{s-1}\right] = L\{e^{3t}+e^{t}\} \qquad e^{-cs}F = L\{u_c f(t-c)\}$$

$$\Rightarrow \quad y(t) = u_2(t)\cdot\{e^{3(t-2)}+e^{t-2}\}$$

b) (*no explanation required, 2 points each*)

True or false:

(i) For the function y you obtained in part (a), $y(1) > 0$. T F

(ii) For the function y you obtained in part (a), $y(3) < 30$. T F

Problem 6. (~~12~~ 14 pts) Solve $ty'' + (t+2)y' + y = -1$, given that $y(0) = 0$.
(Hint: You may use the last formula in the table on the last page of the exam.)

$w = ty$

$w' = ty' + y \quad \Rightarrow \quad ty'' + (t+2)y' + y = -1$

$w'' = ty'' + 2y' \qquad w'' + w' = -1 \qquad w(0) = 0 \cdot y(0) = 0$

$w'(0) = 0 \cdot y'(0) + y(0) = 0$

$w'' + w' = -1$

$\Rightarrow w_H = c_1 + c_2 e^{-t} \quad \Rightarrow w = c_1 + c_2 e^{-t} - t$

$w_p = -t$

$w(0) = c_1 + c_2 - 0 = 0$

$w'(0) = -c_2 - 1 = 0 \Rightarrow c_2 = -1$

$c_1 = +1$

$w = 1 - e^{-t} - t$

$$\boxed{y = \frac{1}{t} - \frac{e^{-t}}{t} - 1}$$

ALTERNATIVELY, LAPLACE TRANSFORM $\quad (-t)f \longrightarrow F' \qquad y(0) = 0 \Rightarrow y'(0) = -\frac{1}{2}$

$\Rightarrow \quad -[s^2F + \frac{1}{2}]' - [sF]' + 2[sF] + F = \frac{-1}{s}$

$\Rightarrow \quad -[s^2F' + 2sF] - [sF' + F] + 2sF + F = \frac{-1}{s}$

$\Rightarrow \quad F' = \frac{1}{s \cdot (s^2 + s)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \quad \Rightarrow \quad (-t)y = -1 + t + e^{-t}$

$$\boxed{y = \frac{1}{t} - 1 - \frac{e^{-t}}{t}}$$

10

Problem 7. a) (~~12~~ pts) Solve the following initial value problem.

$$\begin{cases} x_1' &= -x_1 - x_2 \\ x_2' &= 5x_1 + 3x_2 \end{cases}, \quad x_1(0) = 2, \; x_2(0) = 1$$

$$\vec{X}' = \begin{pmatrix} -1 & -1 \\ 5 & 3 \end{pmatrix} \vec{X} \qquad \vec{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda = ?$ $\det\begin{pmatrix} -1-r & -1 \\ 5 & 3-r \end{pmatrix} = r^2 - 2r - 3 + 5 = r^2 - 2r + 2 \Rightarrow r_1 = 1+i, \; r_2 = 1-i$ (2)

$v_1 = ?$ $\begin{bmatrix} -2-i & -1 \\ 5 & 2-i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} -1 \\ 2+i \end{bmatrix}$ (2)

$$e^t(\cos t + i \sin t)\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \underbrace{\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^t \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \sin t\right)}_{X^{(1)}} + i \underbrace{\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^t \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \cos t\right)}_{X^{(2)}}$$

(3)

$$\Psi(0) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \qquad \vec{X} = \Psi \vec{C} \qquad \vec{X}(0) = \Psi(0)\vec{C} \Rightarrow \vec{C} = \Psi^{-1}(0)\vec{X}(0)$$

$$= \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{X} = -2X^{(1)} + 5X^{(2)}} = \begin{pmatrix} 2e^t \cos t - 5e^t \sin t \\ e^t \cos t + 12 e^t \sin t \end{pmatrix}$$ (3)

b) (*no explanation required, 2 points*) True or false:

The system $\left\{\begin{array}{rcl} x_1' &=& -x_1 - x_2 \\ x_2' &=& 5x_1 + 3x_2 \end{array}\right\}$ is satisfied by the functions $x_1 = e^t \cos t - e^t \sin t$, and $x_2 = -e^t \cos t + 3e^t \sin t$.

T F

$c_1 = -1 \quad c_2 = 1$

Problem 8. (14 pts) Find the general solution of the following system for $t > 0$.

$$\begin{cases} tx_1' &= 2x_1 - x_2 \\ tx_2' &= 3x_1 - 2x_2 \end{cases}$$

(Hint: Look for a solution of the form $x_1 = v_1 t^r$, $x_2 = v_2 t^r$, where v_1, v_2, and r are suitable constants.)

$$\begin{aligned} x_1 &= v_1 t^r \Rightarrow x_1' = r v_1 t^{r-1} \\ x_2 &= v_2 t^r \quad\;\; x_2' = r v_2 t^{r-1} \end{aligned} \Rightarrow \begin{aligned} r v_1 t^r &= 2 v_1 t^r - v_2 t^r \\ r v_2 t^r &= 3 v_1 t^r - 2 v_2 t^r \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} r v_1 &= 2v_1 - v_2 \\ r v_2 &= 3v_1 - 2v_2 \end{aligned} \quad (4) \quad \Rightarrow \quad \underbrace{r}_{\text{eigenvalue}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{eigenvector}} \quad (3)$$

$$\det \begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = 0 \Rightarrow \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \end{aligned} \quad \begin{aligned} r_1 &= 1 \\ r_2 &= -1 \end{aligned} \quad w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; w_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(3)

$$r_1 = 1 \Rightarrow w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= t \\ x_2 &= t \end{aligned} \Rightarrow x^{(1)} = \begin{bmatrix} t \\ t \end{bmatrix}$$

$$r_2 = -1 \Rightarrow w_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= t^{-1} = 1/t \\ x_2 &= 3t^{-1} = 3/t \end{aligned} \quad x^{(2)} = \begin{bmatrix} 1/t \\ 3/t \end{bmatrix}$$

$$X = c_1 x^{(1)} + c_2 x^{(2)} = c_1 \begin{pmatrix} t \\ t \end{pmatrix} + c_2 \begin{pmatrix} 1/t \\ 3/t \end{pmatrix} \quad (4)$$