Math 204: Midterm Exam # 1
Fall 2012

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	KEY

- Please mark the section you are registered below.

  ☐ Section 1 (Mon. & Wed. 12:30-13:45, Instructor: Tolga Etgü)

  ☐ Section 2 (Mon. & Wed. 15:30-16:45, Instructor: Tolga Etgü)

  ☐ Section 3 (Tue. & Thu. 15:30-16:45, Instructor: Barış Coşkunüzer)

- You have 90 minutes.
- *You must show all your work to receive full credit.*

To be filled by the grader:

Problem 1:	
Problem 2:	
Problem 3:	
Problem 4:	
Problem 5:	
Total Grade:	

Problem 1. Find all solutions of the following equations.

a) (10 pts.) $$t^2y' + 3ty = e^{2t}$$

$$y' + \frac{3}{t}y = \frac{e^{2t}}{t^2}$$

$$\mu = e^{\int \frac{3}{t}} = e^{3\ln t} = t^3$$

$$\Rightarrow \quad t^3y' + 3t^2y = te^{2t}$$

$$[t^3y]' = te^{2t}$$

$$t^3y = \int te^{2t}$$

$$\int te^{2t} = \frac{t}{2}e^{2t} - \int \frac{e^{2t}}{2}dt = \frac{t}{2}e^{2t} - \frac{e^{2t}}{4}$$

$u = t \quad dv = e^{2t}dt$

$du = dt \quad v = \frac{e^{2t}}{2}$

$$= \left(\frac{2t-1}{4}\right)e^{2t} + C$$

$$\Rightarrow \quad y = \frac{1}{t^3}\left(\frac{2t-1}{4}\right)e^{2t} + \frac{C}{t^3}$$

$$\boxed{y = \frac{2t-1}{4t^3}e^{2t} + \frac{C}{t^3}}$$

b) (10 pts.) $$x^2 + y^2y' = 3$$

$$y' = \frac{3-x^2}{y^2} \quad \text{(separable)}$$

$$\frac{dy}{dx} = \frac{3-x^2}{y^2} \Rightarrow y^2dy = (3-x^2)dx \Rightarrow \int y^2dy = \int 3-x^2dx$$

$$\Rightarrow \quad \frac{y^3}{3} = 3x - \frac{x^3}{3} + C \quad \Rightarrow \quad \boxed{y^3 = 9x - x^3 + C'}$$

Problem 2.

a) How many solutions does the following initial-value problem have? (6 pts.)

Justify your answer without referring to part (b).

$$\left(\frac{1}{y} - \frac{2}{x^2} + 6x\right) dx + \left(3 - \frac{x}{y^2}\right) dy = 0\,, \quad y(-1) = -\frac{1}{3}$$

Both $y' = f(x,y) = -\dfrac{\frac{1}{y} - \frac{2}{x^2} + 6x}{3 - \frac{x}{y^2}} = -\dfrac{x^2y - 2y^2 + 6x^3y^2}{x^2(3y^2 - x)}$

and $\dfrac{\partial f}{\partial y} = -\dfrac{(x^2 - 4y + 12x^3y)x^2(3y^2-x) - (x^2y - 2y^2 + 6x^3y^2)x^2 6y}{x^4(3y^2-x)^2}$

are continuous on $R = (-\sqrt{3}, 0) \times (-1, 0)$ which contains $(-1, -\frac{1}{3})$. Therefore this IVP has a unique solution by the Existence and Uniqueness Thm.

b) Solve the initial-value problem above. (14 pts.)

$M = \frac{1}{y} - \frac{2}{x^2} + 6x \Rightarrow M_y = \frac{-1}{y^2}$

$M_y = N_x$ exact ✓

$N = 3 - \frac{x}{y^2} \Rightarrow N_x = \frac{-1}{y^2}$

$\Psi_x = \frac{1}{y} - \frac{2}{x^2} + 6x \Rightarrow \Psi = \frac{x}{y} + \frac{2}{x} + 3x^2 + g(y)$

$\Psi_y = \frac{-x}{y^2} + g'(y) = 3 - \frac{x}{y^2} \Rightarrow g'(y) = 3$

$g(y) = 3y$

$\Rightarrow \Psi = \frac{x}{y} + \frac{2}{x} + 3x^2 + 3y$

Solution: $\frac{x}{y} + \frac{2}{x} + 3x^2 + 3y = C \Rightarrow \boxed{\frac{x}{y} + \frac{2}{x} + 3x^2 + 3y = 3}$

$(-1, -\frac{1}{3}) \Rightarrow \frac{-1}{-\frac{1}{3}} + \frac{2}{-1} + 3 + 3\cdot\frac{-1}{3} = 3 - 2 + 3 - 1 = 3 \Rightarrow C = 3$

Problem 3. a) Find all solutions of the following equation. (6 pts.)

$$y'' - 4y' + 4y = 0$$

$y = e^{rt}$ $\quad r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r_1 = r_2 = 2$

$y_1 = e^{2t}$

$y_2 = te^{2t}$

$$\boxed{y = c_1 e^{2t} + c_2 t e^{2t}}$$

b) Find all solutions of the following equation. (14 pts.)

$$y'' - 4y' + 4y = \frac{-e^{2t}}{t}$$

$y_p = u y_1 + v y_2$ $\quad u = -\int \frac{y_2 \cdot g}{W}$ $\quad v = \int \frac{y_1 g}{W}$

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = (2t+1)e^{4t} - 2te^{4t} = e^{4t}$$

$$u = -\int \frac{te^{2t} \cdot \frac{-e^{2t}}{t}}{e^{4t}} = \int dt = t + C_1$$

$$v = \int \frac{e^{2t} \cdot \frac{-e^{2t}}{t}}{e^{4t}} = -\int \frac{1}{t}\, dt = -\ln t + C_2$$

$\Rightarrow$ $y_p = te^{2t} - \ln t \cdot te^{2t}$

$$y = c_1 e^{2t} + c_2 te^{2t} + te^{2t} - \ln t \cdot t \cdot e^{2t}$$

Problem 4. Find all solutions of the following equation. (25 pts.)

(*Hint: Look for a solution of the form* $y = t^r \sin t$)

$$ty'' + 2y' + ty = 0$$

ALTERNATIVELY:

$w = ty$

$w' = y + ty'$ $\Rightarrow$ $ty'' + 2y' + ty = 0$

$w'' = 2y' + ty''$

$w'' + w = 0$

$w_1 = \sin t \quad w_2 = \cos t$

$y_1 = \frac{\sin t}{t} \quad y_2 = \frac{\cos t}{t}$

$y = t^r \sin t$

$y' = r t^{r-1} \sin t + t^r \cos t$

$y'' = r(r-1) t^{r-2} \sin t + 2r t^{r-1} \cos t - t^r \sin t$

$ty'' + 2y' + ty = 0 \Rightarrow$

$[(r^2 - r) t^{r-1} \sin t + 2r t^r \cos t - \cancel{t^{r+1} \sin t}] + (2r t^{r-1} \sin t + 2t^r \cos t) + \cancel{t^{r+1} \sin t} = 0$

$(r^2 + r) t^{r-1} \sin t + (2r + 2) t^r \cos t = 0 \quad \Rightarrow \quad r = -1 \quad \Rightarrow \quad y_1 = t^{-1} \sin t = \frac{\sin t}{t}$

$y_2 = ?$

REDUCTION of ORDER

$y_2 = u y_1$

$y_2' = u' y_1 + u y_1'$

$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$

$\Rightarrow \quad t(u'' y_1 + 2u' y_1' + u y_1'') + 2(u' y_1 + u y_1') + t u y_1 = 0$

$t y_1 u'' + (2t y_1' + 2y_1) u' + \underbrace{(t y_1'' + 2y_1' + t y_1)}_{=0} u = 0$

$\Rightarrow \quad t y_1 u'' + (2t y_1' + 2y_1) u' = 0 \qquad y_1 = \frac{\sin t}{t} \qquad y_1' = \frac{\cos t \cdot t - \sin t}{t^2} \qquad v = u'$

$\Rightarrow \quad t \cdot \frac{\sin t}{t} \cdot v' + \left(2t \cdot \frac{\cos t \cdot t - \sin t}{t^2} + 2\frac{\sin t}{t}\right) v = 0$

$\Rightarrow \quad \sin t \, v' + 2\cos t \, v = 0 \qquad \mu: \sin t$

$\sin^2 t \, v' + 2 \sin t \cos t \, v = 0$

$(\sin^2 t \, v)' = 0 \quad \Rightarrow \quad v = \frac{1}{\sin^2 t} \quad \Rightarrow \quad u' = \frac{1}{\sin^2 t} \quad \Rightarrow \quad u = -\cot t$

$\Rightarrow y_2 = -\frac{\cos t}{\sin t} \cdot \frac{\sin t}{t} = \frac{-\cos t}{t} \quad \Rightarrow \quad \boxed{y = c_1 \frac{\sin t}{t} + c_2 \frac{\cos t}{t}}$

Problem 5. Suppose that y_1 and y_2 are solutions of (15 pts.)

$$y'' + p(t)y' + q(t)y = 0$$

on an open interval I on which both p and q are continuous. Prove that if there is a point t_0 in I such that $y_1(t_0) = y_2(t_0) = 0$, then there are solutions which are not of the form $c_1y_1 + c_2y_2$.

$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = 0 \cdot y_2'(t_0) - 0 \cdot y_1'(t_0) = 0$

$\Rightarrow$ $\{y_1, y_2\}$ is not a fundamental set for the equation.

By the theorem any linear 2nd order DE has a fund. set $\{f_1, f_2\}$.
s.t. $y = c_1 f_1 + c_2 f_2$ is general solution.

Hence, there must be another solution $y = f$ s.t. $\{y_1, f\}$ is fundamental set, and so $f \neq c_1y_1 + c_2y_2$.

ALTERNATIVE SOLUTION: By the Existence and Uniqueness Thm., the IVP
$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = 1 \\ y'(t_0) = 0 \end{cases}$$
has a unique solution, call it y_3. Since $y_3(t_0) = 1$ there are no constants c_1 and c_2 such that $y_3 = c_1y_1 + c_2y_2$ (otherwise we would have $y_3(t_0) = c_1y_1(t_0) + c_2y_2(t_0) = 0$)