Problem 1. a) (8 pts) Find the Laplace transform of e^{6t} using only the definition.

$$\mathcal{L}\{e^{6t}\} = \int_0^\infty e^{-st}e^{6t}\,dt = \lim_{a\to\infty}\int_0^a e^{(6-s)t}\,dt = \lim_{a\to\infty}\left[\left.\frac{e^{(6-s)t}}{6-s}\right|_0^a\right]$$

$$= \lim_{a\to\infty}\left[\frac{e^{(6-s)a}}{6-s} - \frac{1}{6-s}\right]$$

$*$ In order to have $\lim_{a\to\infty}\frac{e^{(6-s)a}}{6-s}$ as finite, we must have $6-s<0$

$$= \frac{1}{s-6}, \quad s>6$$

b) (10 pts.) Determine the function whose Laplace transform is $e^{-s}\frac{s-1}{s^2(s^2+4)}$.
(You can use the table on the last page of the exam.)

$$\frac{s-1}{s^2(s^2+4)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4} = \frac{(A+C)s^3+(B+D)s^2+4As+4B}{s^2(s^2+4)} \Rightarrow \begin{array}{l} A=\frac{1}{4},\ B=-\frac{1}{4} \\ C=-\frac{1}{4},\ D=\frac{1}{4}\end{array}$$

Denote $\frac{s-1}{s^2(s^2+4)} =: F(s)$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\cdot\frac{s}{s^2+4} + \frac{1}{8}\cdot\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{4} - \frac{1}{4}t - \frac{1}{4}\cos 2t + \frac{1}{8}\sin 2t$$

Then,

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{s-1}{s^2(s^2+4)}\right\} = u_1(t)\cdot f(t-1)$$

$$= u_1(t)\left[\frac{1}{4} - \frac{1}{4}(t-1) - \frac{1}{4}\cos(2t-2) + \frac{1}{8}\sin(2t-2)\right]$$

c) (*no explanation required, 4 points if both of your answers are correct, 0 otherwise*)
True or false:

(i) The Laplace transform of e^{6t} exists only for $s>6$. T F

(ii) The Laplace transform of $\sin t\cos t$ is $\frac{s}{(s^2+1)^2}$. T F

Problem 2. a) (16 pts) Find the solution of the following initial value problem.

$$y'' + 4y = \begin{cases} 4t & , \quad t < 1 \\ 4 & , \quad t \geq 1 \end{cases}$$

$$y(0) = 0, \; y'(0) = 1$$

$g(t) = 4t(1 - u_1(t)) + 4 . u_1(t) = 4t - 4u_1(t)(t-1) \to (3)$

$L(g(t)) = \frac{4}{s^2} - \frac{4e^{-s}}{s^2} \to (3)$

$[s^2 F(s) - s.0 - 1] + 4.F(s) = L(g(t)) \Rightarrow (s^2+4)F(s) = 1 + \frac{4}{s^2} - \frac{4}{s^2}e^{-s}$

$\Rightarrow F(s) = \frac{1}{s^2+4} + \frac{4}{s^2.(s^2+4)} - \frac{4.e^{-s}}{s^2.(s^2+4)} \to (3)$

$\frac{4}{s^2.(s^2+4)} = \frac{1}{s^2} - \frac{1}{s^2+4} \Rightarrow F(s) = \cancel{\frac{1}{s^2+4}} + \left[\frac{1}{s^2} - \frac{1}{s^2+4}\right] - e^{-s}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right] \to (4)$

$\Rightarrow \boxed{f(t) = t - u_1(t)\left[(t-1) - \frac{1}{2}\sin 2(t-1)\right]} \to (3)$

b) (*no explanation required, 4 points if both of your answers are correct, 0 otherwise*)
True or false:

(i) For the solution y of the initial value problem above, $y(\frac{1}{2}) = \frac{1}{2}$. T F

(ii) For the solution y of the initial value problem above, $y(t) < 4$ for every $t \geq 1$. T F

Problem 3. a) (12 pts) Assuming that $g(t)$ is a piecewise-continuous function of exponential order, express the solution of the following initial value problem in terms of a convolution integral.

$$y'' + 2y' + y = g(t), \ y(0) = 0, \ y'(0) = 1$$

Let $Y = \mathcal{L}\{y\}$ and $G = \mathcal{L}\{g\}$. Then

$$[s^2 Y - sy(0) - y'(0)] + 2[sY - 2y(0)] + Y = G$$

So

$$Y = \frac{1}{(s+1)^2} + G \cdot \frac{1}{(s+1)^2}$$

Hence

$$y = \mathcal{L}^{-1}\{Y\} = te^{-t} + g(t) * te^{-t}$$

$$= \boxed{te^{-t} + \int_0^t g(t-\tau)\,\tau e^{-\tau}\,d\tau}$$

b) (5 pts) Calculate $y(2)$, where y is the solution of the initial value problem above for

$$g(t) = e^{2-t}$$

$$y(t) = te^{-t} + \int_0^t e^{2-(t-\tau)}\,\tau e^{-\tau}\,d\tau$$

$$y(2) = 2e^{-2} + \int_0^2 e^{2-(2-\tau)}\,\tau e^{-\tau}\,d\tau$$

$$= 2e^{-2} + \int_0^2 \tau\,d\tau = \boxed{2e^{-2} + 2}$$

Problem 4. a) (18 pts) Find the general solution of the following system.

$$\begin{cases} x_1' &= x_1 - x_2 + x_3 \\ x_2' &= 5x_1 - 3x_2 + x_3 \\ x_3' &= x_3 \end{cases}$$

$x' = Ax$, where $A = \begin{pmatrix} 1 & -1 & 1 \\ 5 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Let's first find the eigenvalues, i.e., the roots of the equation $\det(A - rI) = 0$.

$$\begin{vmatrix} 1-r & -1 & 1 \\ 5 & -3-r & 1 \\ 0 & 0 & 1-r \end{vmatrix} = (1-r)\left[(1-r)(-3-r)+5\right] = 0$$

$$\Rightarrow (1-r)\left[-3-r+3r+r^2+5\right] = 0 \Rightarrow (1-r)\left[r^2+2r+2\right] = 0$$

$$\Rightarrow r = 1, \quad r = \frac{-2+\sqrt{4-8}}{2} = -1+i, \quad r = \frac{-2-\sqrt{4-8}}{2} = -1-i$$

Let's find the eigenvector corresponding to $r = 1$.

$$\begin{pmatrix} 0 & -1 & 1 \\ 5 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(1)} = \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$$

b) (*no explanation required, 4 points if both of your answers are correct, 0 otherwise*)

True or false:

(i) $x_1 = \cos t,\ x_2 = 2\cos t + \sin t,\ x_3 = 5e^t$ is a solution of the system above. T F

(ii) $x_1 = 3e^t + 2\cos t,\ x_2 = 5e^t + 4\cos t + 2\sin t,\ x_3 = 5e^t$ is a solution of the system above. T F

Now let's find the eigenvector corresponding to $r=-1+i$.

$$\begin{pmatrix} 2-i & -1 & 1 \\ 5 & -2-i & 1 \\ 0 & 0 & 2-i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 2-i \\ 0 \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 1 \\ 2-i \\ 0 \end{pmatrix} e^{(-1+i)t} = \begin{pmatrix} 1 \\ 2-i \\ 0 \end{pmatrix} e^{-t} e^{it} = e^{-t} \begin{pmatrix} 1 \\ 2-i \\ 0 \end{pmatrix} (\cos t + i \sin t)$$

$$= e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \\ 0 \end{pmatrix} + i e^{-t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \\ 0 \end{pmatrix}$$

Hence the general solution is

$$x = c_1 \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \\ 0 \end{pmatrix} e^{-t}$$

for some scalars c_1, c_2, c_3.

Problem 5. a) (15 pts) Find a fundamental matrix for the system

$$\begin{cases} x_1' &= 3x_1 + x_2 \\ x_2' &= 2x_1 + 4x_2 \end{cases}$$

and solve the following initial value problem.

$$\begin{cases} x_1' &= 3x_1 + x_2 \\ x_2' &= 2x_1 + 4x_2 \end{cases}, \quad x_1(0) = 2,\ x_2(0) = 1$$

The system is $x' = A \cdot x$, where $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

$0 = \det(A - rI) = \begin{vmatrix} 3-r & 1 \\ 2 & 4-r \end{vmatrix} = (r-5)(r-2)$

So the eigenvalues are $r=2$ and $r=5$

eigenvectors for $r=2$: $(A-2I)\cdot v = 0$

$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 = -v_2$

$\boxed{x^{(1)} = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ is a solution.

eigenvectors for $r=5$: $(A-5I)\cdot v = 0$

$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2v_1 = v_2$

$\boxed{x^{(2)} = e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ is a solution.

A fundamental matrix is $\boxed{\Psi(t) = \begin{pmatrix} e^{2t} & e^{5t} \\ -e^{2t} & 2e^{5t} \end{pmatrix}}$

Unique solution of the IVP is

$x = \Psi(t)\cdot(\Psi(0))^{-1}\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} e^{2t}+e^{5t} \\ -e^{2t}+2e^{5t} \end{pmatrix}}$

b) (*no explanation required, 4 points if both of your answers are correct, 0 otherwise*)

True or false:

(i) $x_1 = 2e^{5t} - e^{2t}$, $x_2 = 4e^{5t} - e^{2t}$ is a solution of the system $\begin{cases} x_1' &= 3x_1 + x_2 \\ x_2' &= 2x_1 + 4x_2 \end{cases}$ **T** **F**

(ii) $\begin{pmatrix} e^{5t} & 2e^{5t} \\ 2e^{5t} & 4e^{5t} \end{pmatrix}$ is a fundamental matrix of this system. **T** **F**