Problem 1.a) (10 pts) Find the general solution of the following differential equation.

$$D^2(D^2+1)(D^2-1)y = 0$$

$r_1 = 0$, $r_2 = i$, $r_3 = -i$, $r_4 = 1$, $r_5 = -1$

where $r_1 = 0$ is a double root.

Hence the general solution is

$$y(t) = c_1 + c_2 t + c_3 \cos t + c_4 \sin t + c_5 e^t + c_6 e^{-t},$$

c_i constants.

---

b) (15 pts.) Determine a suitable form for a particular solution of the following differential equation. (Do not calculate the coefficients.)

$$D^2(D^2+1)(D^2-1)y = 2t^3 - 5e^{-t} - \cos t + 4e^t \sin t$$

D^4, $(D+1)$, (D^2+1) and $((D-1)^2+1)$ are annihilators of $2t^3$, $-5e^{-t}$, $-\cos t$ and $4e^t \sin t$, respectively.

Hence

$$D^2(D^2+1)(D^2-1)(D^4)(D+1)(D^2+1)((D-1)^2+1)\,y = 0$$

$$\Rightarrow D^6(D^2+1)^2(D-1)(D+1)^2(D^2-2D+2)\,y = 0$$

Hence $y = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_6 t^5 + c_7 \cos t$

$+ c_8 \sin t + c_9 t \sin t + c_{10} t \cos t + c_{11} e^t$

$+ c_{12} e^{-t} + c_{13} t e^{-t} + c_{14} e^t \sin t + c_{15} e^t \cos t$

$\Rightarrow Y_p = c_3 t^2 + c_4 t^3 + c_5 t^4 + c_6 t^5 + c_9 t \sin t + c_{10} t \cos t$

$+ c_{13} t e^{-t} + c_{14} e^t \sin t + c_{15} e^t \cos t$, where c_i are constants.

Problem 2. (20 pts) Find the general solution of the following differential equation.

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4 \quad , \quad x > 0$$

(Hint: x, x^2, x^{-1} are solutions for $x^3y''' + x^2y'' - 2xy' + 2y = 0$)

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 2x$$

$$W = \begin{vmatrix} x & x^2 & \frac{1}{x} \\ 1 & 2x & -1/x^2 \\ 0 & 2 & \frac{2}{x^3} \end{vmatrix} = x\left(\frac{4}{x^2} + \frac{2}{x^2}\right) - 1\left(\frac{2}{x} - \frac{2}{x}\right) = \frac{6}{x}$$ ✓

$$y_p = u_1y_1 + u_2y_2 + u_3y_3 \qquad u_k' = \frac{g \cdot W_k}{W}$$

$$W_1 = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = 1 \cdot (-1-2) = -3 \qquad W_2 = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = \frac{+2}{x}$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2x^2 - x^2 = x^2 \Rightarrow$$

$$u_1' = \frac{2x \cdot -3}{\frac{6}{x}} = -x^2 \Rightarrow u_1 = \frac{-x^3}{3}$$

$$u_2' = \frac{2x \cdot \frac{2}{x}}{\frac{6}{x}} = \frac{2x}{3} \Rightarrow u_2 = \frac{x^2}{3}$$

$$u_3' = \frac{2x \cdot x^2}{\frac{6}{x}} = \frac{x^4}{3} \Rightarrow u_3 = \frac{x^5}{15}$$

ALTERNATIVELY,

TRY: $y_p = Ax^4$ (Euler Equation)

$\Rightarrow A = \frac{1}{15}$

$$\Rightarrow y_p = x \cdot \frac{-x^3}{3} + x^2 \cdot \frac{x^2}{3} + \frac{1}{x} \cdot \frac{x^5}{15} = \boxed{\frac{x^4}{15}}$$

$$\boxed{y = c_1x + c_2x^2 + \frac{c_3}{x} + \frac{x^4}{15}}$$

Problem 3. a) (12 pts) Determine the first 4 terms of the power series solution of the following initial value problem.

$$(x^2+4x+5)y'' + 2xy' - 3y = 0 \quad , \qquad y(0) = 5 \ , \ y'(0) = 2$$

$y = \phi(x) = \sum_{n=0}^{\infty} a_n x^n \qquad a_n = \dfrac{\phi^{(n)}(0)}{n!} \qquad \phi(0) = 5 \quad \phi'(0) = 2.$

$a_0 = \boxed{5} \quad a_1 = \boxed{2}$

$a_2 = \dfrac{\phi''(0)}{2!} \qquad (0+0+5)\phi''(0) + 2\cdot 0\,\phi'(0) - 3\phi(0) = 0 \Rightarrow \phi''(0) = \dfrac{3}{5}\cdot 5 = 3.$

$\Rightarrow a_2 = \boxed{\dfrac{3}{2}}$

$(2x+4)\phi'' + (x^2+4x+5)\phi''' + 2\phi' + 2x\phi'' - 3\phi' = 0$

$a_3 = \dfrac{\phi'''(0)}{3!} \qquad 4\phi''(0) + 5\phi'''(0) + 2\phi'(0) - 3\phi'(0) = 0$

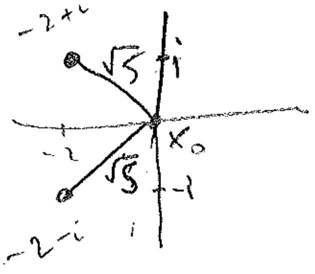
$= \dfrac{-2}{6} = \boxed{\dfrac{-1}{3}} \qquad 4\cdot 3 + 5\cdot\phi'''(0) + 2\cdot 2 - 3\cdot 2 = 0 \Rightarrow \phi'''(0) = \dfrac{-10}{5} = -2$

$y = 5 + 2x + \dfrac{3}{2}x^2 - \dfrac{1}{3}x^3 + \cdots$

b) (8 pts) Determine a lower bound for the radius of convergence of the power series in part (a).

$$y'' + \frac{2x}{x^2+4x+5}y' - \frac{3}{x^2+4x+5}y = 0$$

$x_0 = 0 \qquad x^2+4x+5 = 0 \Rightarrow \begin{array}{l} x = -2+i \\ x = -2-i \end{array}$



$\Rightarrow \quad \rho \geq \sqrt{5}$

Problem 4. (20 pts) Find the general terms of two linearly independent power series solutions of the following differential equation about the point $x_0 = 0$.

$$y'' + xy' + 2y = 0$$

$y = \sum a_n x^n$

$y' = \sum n a_n x^{n-1}$

$y'' = \sum n(n-1) a_n x^{n-2}$

$\Rightarrow \sum n(n-1) a_n x^{n-2} + x \sum n a_n x^{n-1} + 2 \sum a_n x^n = 0$

$\sum_{2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{0}^{\infty} (n a_n + 2a_n) x^n = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum (n+2) a_n x^n = 0$

$\Rightarrow (n+2)(n+1) a_{n+2} + (n+2) a_n = 0 \Rightarrow \boxed{a_{n+2} = \frac{-a_n}{n+1}}$ 12

$\Rightarrow a_2 = \frac{-a_0}{1}$ | $a_3 = \frac{-a_1}{2}$

$a_4 = \frac{-a_2}{3} = \frac{a_0}{1.3}$ | $a_5 = \frac{-a_3}{4} = \frac{a_1}{2.4}$

$a_6 = \frac{-a_4}{5} = \frac{-a_0}{1.3.5}$ | $a_7 = \frac{-a_5}{6} = \frac{-a_1}{2.4.6}$

$a_{2k} = \frac{(-1)^k a_0}{1.3. \ (2k-1)}$ $k \geq 1$ | $a_{2k+1} = \frac{(-1)^k a_1}{2.4.6.. 2k}$ / 5

$\Rightarrow y = \sum a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} = \underbrace{a_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{1.3.5.(2k-1)} x^{2k}\right)}_{y_1} + \underbrace{a_1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2.4.6.2k} x^{2k+1}\right)}_{y_2}$

3

Problem 5. (15 pts) Solve the following initial value problem.

$$4x^2y'' + 8xy' + 17y = 0 \quad , \qquad y(1) = 4 \ , \ y'(1) = 6$$

$y = x^r \Rightarrow \quad 4x^2 r.(r-1)x^{r-2} + 8x r . x^{r-1} + 17x^r = 0$

$(4r^2 - 4r + 8r + 17)x^r = 0$

$4r^2 + 4r + 17 = 0 \Rightarrow \quad 4r^2 + 4r + 1 + 16 = 0$

$r^2 + r + \frac{1}{4} + 4 = 0$

$(r + \frac{1}{2})^2 + 2^2 = 0$

$r = -\frac{1}{2} \pm 2i$

$\Rightarrow \quad y_1 = x^{-1/2}\cos(2\ln x) \quad y_2 = x^{-1/2}\sin(2\ln x)$

$y = c_1 y_1 + c_2 y_2$

$y(1) = 4 = c_1 \cdot 1 \cdot \cos 0 + c_2 \cdot 1 \cdot \sin 0 \Rightarrow c_1 = \boxed{4}$

$y'(1) = 6 \quad y' = \frac{-1}{2}c_1 x^{-3/2} \cdot \cos(2\ln x) - c_1 x^{-1/2}\sin(2\ln x) \cdot \frac{2}{x} + \frac{-1}{2}x^{-3/2}c_2 \sin(2\ln x)$
$+ x^{-1/2}c_2\cos(2\ln x) \cdot \frac{2}{x}$

$y'(1) = \frac{-1}{2} \cdot 1 \cdot c_1 - 0 - 0 + 1 \cdot 1 \cdot 2c_2$

$6 = \frac{-4}{2} + 2c_2 \Rightarrow c_2 = \boxed{4}$

$y = \frac{4}{\sqrt{x}}\cos(2\ln x) + \frac{4}{\sqrt{x}}\sin(2\ln x)$