

REVIEW QUESTIONS for FINAL

6.3

15. The function can be expressed as

$$f(t) = (t - \pi) [u_\pi(t) - u_{2\pi}(t)].$$

Before invoking the translation property of the transform, write the function as

$$f(t) = (t - \pi) u_\pi(t) - (t - 2\pi) u_{2\pi}(t) - \pi u_{2\pi}(t).$$

It follows that

$$\mathcal{L}[f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

21. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2e^t \cos t.$$

Hence

$$\mathcal{L}^{-1}[e^{-2s}G(s)] = 2e^{t-2} \cos(t-2) u_2(t).$$

6.4.

2.(a) Let $h(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 2sY(s) + 2Y(s) - 1 = \mathcal{L}[h(t)].$$

The forcing function can be written as $h(t) = 2(u_\pi(t) - u_{2\pi}(t))$. Its transform is

$$\mathcal{L}[h(t)] = \frac{2(e^{-\pi s} - e^{-2\pi s})}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{2(e^{-\pi s} - e^{-2\pi s})}{s(s^2 + 2s + 2)}.$$

First note that

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}.$$

Using partial fractions,

$$\frac{2}{s(s^2 + 2s + 2)} = \frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 1}.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}^{-1} \left[\frac{1}{s^2 + 2s + 2} \right] = \mathcal{L}^{-1} \left[\frac{1}{(s + 1)^2 + 1} \right] = e^{-t} \sin t.$$

Now let

$$G(s) = \frac{2}{s(s^2 + 2s + 2)}.$$

Then

$$\mathcal{L}^{-1} [G(s)] = 1 - e^{-t} \cos t - e^{-t} \sin t.$$

Using Theorem 6.3.1,

$$\mathcal{L}^{-1} [e^{-cs} G(s)] = u_c(t) - e^{-(t-c)} [\cos(t-c) + \sin(t-c)] u_c(t).$$

Hence the solution of the IVP is

$$y(t) = e^{-t} \sin t + u_\pi(t) - e^{-(t-\pi)} [\cos(t-\pi) + \sin(t-\pi)] u_\pi(t) - u_{2\pi}(t) + e^{-(t-2\pi)} [\cos(t-2\pi) + \sin(t-2\pi)] u_{2\pi}(t).$$

That is,

$$y(t) = e^{-t} \sin t + [u_\pi(t) - u_{2\pi}(t)] + e^{-(t-\pi)} [\cos t + \sin t] u_\pi(t) + e^{-(t-2\pi)} [\cos t + \sin t] u_{2\pi}(t).$$

12.(a) Taking the Laplace transform of the ODE, we obtain

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) + \\ + 5 [s^2 Y(s) - s y(0) - y'(0)] + 4 Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Applying the initial conditions,

$$s^4 Y(s) + 5s^2 Y(s) + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s(s^4 + 5s^2 + 4)} - \frac{e^{-\pi s}}{s(s^4 + 5s^2 + 4)}.$$

Using partial fractions,

$$\frac{1}{s(s^4 + 5s^2 + 4)} = \frac{1}{12} \left[\frac{3}{s} + \frac{s}{s^2 + 4} - \frac{4s}{s^2 + 1} \right].$$

It follows that

$$\mathcal{L}^{-1} \left[\frac{1}{s(s^4 + 5s^2 + 4)} \right] = \frac{1}{12} [3 + \cos 2t - 4 \cos t].$$

Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{4} [1 - u_\pi(t)] + \frac{1}{12} [\cos 2t - 4 \cos t] - \\ - \frac{1}{12} [\cos 2(t - \pi) - 4 \cos(t - \pi)] u_\pi(t).$$

That is,

$$y(t) = \frac{1}{4} [1 - u_\pi(t)] + \frac{1}{12} [\cos 2t - 4 \cos t] - \\ - \frac{1}{12} [\cos 2t + 4 \cos t] u_\pi(t).$$

6.6

6. We have $\mathcal{L}[e^{-t}] = 1/(s+1)$ and $\mathcal{L}[\sin t] = 1/(s^2+1)$. Based on Theorem 6.6.1,

$$\mathcal{L} \left[\int_0^t e^{-(t-\tau)} \sin(\tau) d\tau \right] = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}.$$

13. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) - 1 + \omega^2 Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2 + \omega^2} + \frac{G(s)}{s^2 + \omega^2}.$$

As shown in a related situation, Problem 11,

$$\mathcal{L}^{-1} \left[\frac{G(s)}{s^2 + \omega^2} \right] = \frac{1}{\omega} \int_0^t \sin(\omega(t - \tau)) g(\tau) d\tau.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{\omega} \sin(\omega t) + \frac{1}{\omega} \int_0^t \sin(\omega(t - \tau)) g(\tau) d\tau.$$

7.1

3. First divide both sides of the equation by t^2 , and write

$$u'' = -\frac{1}{t} u' - \left(1 - \frac{1}{t^2}\right)u.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$x_2' = u'' = -\frac{1}{t} u' - \left(1 - \frac{1}{t^2}\right)u.$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\left(1 - \frac{1}{t^2}\right)x_1 - \frac{1}{t} x_2. \end{aligned}$$

7.2

11. First augment the given matrix by the identity matrix:

$$[\mathbf{A} | \mathbf{I}] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Divide the first row by 3, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.$$

Adding -6 times the first row to the second row results in

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.$$

Divide the second row by 4, to obtain

$$\begin{pmatrix} 1 & -1/3 & 1/3 & 0 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Finally, adding $1/3$ times the second row to the first row results in

$$\begin{pmatrix} 1 & 0 & 1/6 & 1/12 \\ 0 & 1 & -1/2 & 1/4 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}^{-1} = \frac{1}{12} \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}.$$

13. Elementary row operations yield

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}$$

Finally, combining the first and third rows results in

$$\begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{pmatrix}, \text{ so } A^{-1} = \begin{pmatrix} 1/2 & -1/4 & 1/8 \\ 0 & 1/2 & -1/4 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

7.3

4. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array} \right).$$

Adding -2 times the first row to the second row and subtracting the first row from the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right).$$

Adding the negative of the second row to the third row results in

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

We evidently end up with an equivalent system of equations

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0.\end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$x = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

8. Write the given vectors as columns of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $\det(\mathbf{X}) = 0$. Hence the vectors are linearly dependent. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

We obtain the system of equations

$$\begin{aligned}c_1 - 3c_3 &= 0 \\ c_2 + 5c_3 &= 0.\end{aligned}$$

Setting $c_3 = 1$, it follows that $c_1 = 3$ and $c_2 = -5$. Hence

$$3\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + \mathbf{x}^{(3)} = \mathbf{0}.$$

7.5

3.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -3$. For $r = 2$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. Substitution of $r = -3$ results in the single equation $4\xi_1 + \xi_2 = 0$. A corresponding

eigenvector is $\xi^{(2)} = (1, -4)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

12. Setting $\mathbf{x} = \xi e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\xi^{(1)} = (1, 1, 1)^T$. Setting $\lambda = 1$, the reduced system of equations is

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\xi_3 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\xi^{(2)} = (1, -2, 1)^T$. Finally, setting $\lambda = -1$, the reduced system of equations is

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0. \end{aligned}$$

A corresponding solution vector is given by $\xi^{(3)} = (1, 0, -1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

15. Setting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 3)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 4 \\ c_1 + 3c_2 &= -2. \end{aligned}$$

Hence $c_1 = 7$ and $c_2 = -3$, and the solution of the IVP is

$$\mathbf{x} = 7 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - 3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

7.6

2.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i, 1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i, 1)^T$. Hence one of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (\cos t + i \sin t) = \\ &= \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ -\cos t + 2 \sin t \end{pmatrix}.$$

10. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 4r + 5 = 0$, with roots $r = -2 \pm i$. Substituting $r = -2 + i$, the equations are equivalent to $\xi_1 - (1 - i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1 - i, 1)^T$. One of the complex-valued solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{-2t} (\cos t + i \sin t) = \\ &= e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

Hence the general solution is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 - c_2 &= 2 \\ c_1 &= -4. \end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

$$\begin{aligned} \mathbf{x} &= -4 e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} - 6 e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix} \\ &= 2 e^{-2t} \begin{pmatrix} \cos t - 5 \sin t \\ -2 \cos t - 3 \sin t \end{pmatrix}. \end{aligned}$$

7.7

3.(a,b) The general solution, found in Problem 3, Section 7.6, is given by

$$\mathbf{x} = c_1 \begin{pmatrix} 5 \cos t \\ 2 \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5 \sin t \\ \cos t + 2 \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 1 \\ 2c_1 + c_2 &= 0, \end{aligned}$$

resulting in $c_1 = 1/5$, $c_2 = -2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 0 \\ 2c_1 - c_2 &= 1, \end{aligned}$$

resulting in $c_1 = 0$, $c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -5 \sin t \\ \cos t - 2 \sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\Phi(t) = \begin{pmatrix} \cos t + 2 \sin t & -5 \sin t \\ \sin t & \cos t - 2 \sin t \end{pmatrix}.$$

7.8

2 (c) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -3 - r & \frac{5}{2} \\ -\frac{5}{2} & 2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\mathbf{I}) = r^2 + r + \frac{1}{4} = 0$. The only root is $r = -1/2$, which is an eigenvalue of multiplicity two. Setting $r = -1/2$ the coefficient matrix reduces the system to the single equation $-\xi_1 + \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}.$$

In order to obtain a second linearly independent solution, we find a solution of the system

$$\begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These equations reduce to $-5\eta_1 + 5\eta_2 = 2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = k + 2/5$. A second solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} k \\ k + 2/5 \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} \right].$$

7.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2} - r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -3/2 & 3/2 \\ -3/2 & 3/2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} k \\ 2/3 + k \end{pmatrix} e^{-t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} \right].$$

Imposing the initial conditions, find that

$$\begin{aligned} c_1 &= 3 \\ c_1 + \frac{2}{3}c_2 &= -1, \end{aligned}$$

so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} t e^{-t}.$$