2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let $A \subset (0, 1)$ has infinitely many elements. Then, A has an accumulation point.

TRUE.

Take a sequence $(a_n) \subseteq A$ s.t. $a_n \neq a_m$ if $n \neq m$.

BW $\Rightarrow$ $\exists$ cluster point c of (a_n). c is an accumulation pt of A since $a_n \neq a_m$ for $n \neq m$.

2b) If a sequence (a_n) in $\mathbb{R}$ has a unique cluster point, then (a_n) is convergent.

FALSE.

$a_n = \{1, 1, 1, 2, 1, 3, 1, 4, 1, 5, \dots\}$

only cluster pt is 1.

2c) Every open subset in $\mathbb{R}$ can be written as a union of closed sets.

TRUE. $O \subseteq \mathbb{R}$ $\quad O = \bigcup_{x \in O} \{x\}$

$\forall x \in \mathbb{R}$ $\{x\}$ is closed.

(If $O \neq \emptyset$ $\Rightarrow$ $\emptyset$ is closed, too.)

2d) There is an uncountable set $A \subset \mathbb{R}$ such that $\mathring{A} = \emptyset$

TRUE. $A = \mathbb{R} - \mathbb{Q}$. uncountable

$\mathring{A} = \emptyset$

3) (20 pts) Prove or give a counterexample for the following statement.

Any Cauchy sequence in $\mathbb{Q}$ is convergent.

FALSE.

COUNTEREXAMPLE: Since $\mathbb{Q}$ is dense in $\mathbb{R}$,

$\exists\, (q_n) \subseteq \mathbb{Q}$ s.t. $q_n \longrightarrow \sqrt{2}$ $(\sqrt{2} \notin \mathbb{Q})$.

First, we claim that (q_n) is Cauchy.

Since (q_n) is convergent in $\mathbb{R} \Rightarrow (q_n)$ is also Cauchy in $\mathbb{R}$.

since $\mathbb{Q}$ is subspace of $\mathbb{R}$, (q_n) is Cauchy in $\mathbb{Q}$, too. (same metric)

However, $q_n \to \sqrt{2}$ in $\mathbb{R}$, and the limit is unique. Hence

q_n is not convergent in $\mathbb{Q}$.

4) (20 pts) Prove or give a counterexample for the following statement.

$\limsup a_n = \liminf a_n$ if and only if (a_n) is convergent.

TRUE.

$\Rightarrow$ Let $\limsup a_n = \liminf a_n = L$.

Since $\limsup a_n$ exists, (a_n) bounded above
Since $\liminf a_n$ exists, (a_n) bounded below $\Rightarrow$ (a_n) bounded.

By Theorem, if (a_n) bdd, limsup = largest cluster pt
liminf = smallest cluster pt. (*)

Then, since limsup = liminf $\Rightarrow$ $\exists!$ cluster point L.

Since (a_n) bdd & $\exists$ unique cluster pt $\Rightarrow$ (a_n) convergent.
Theorem.

$\Leftarrow$ (a_n) convergent, then (a_n) bdd, Hence, limsup & liminf exists.

By (*), $\limsup a_n = L$ & $\liminf a_n = L$ since L is the unique cluster pt as (a_n) convergent. $\square$.

5) (20 pts) Prove or give a counterexample for the following statement.

The derived set A' is closed.

TRUE.

$A' = \{\text{accumulation pts of } A\}$

Let $(x_n) \subseteq A'$, and $x_n \to L$. If we can show that $L \in A'$, we are done.

Claim: $L \in A'$, (i.e. $\forall \varepsilon$ $(L-\varepsilon, L+\varepsilon) \cap A \supseteq \{a_\varepsilon\}$ $a_\varepsilon \in A$ and $a_\varepsilon \neq L$)

let $\varepsilon_0 > 0$ be given.

Since $x_n \to L$ $\exists N_0$ s.t. $\forall n \geq N_0$ $|x_n - L| < \frac{\varepsilon_0}{2}$

Since $x_{N_0} \in A'$ $(x_{N_0} - \frac{\varepsilon_0}{2}, x_{N_0} + \frac{\varepsilon_0}{2})$ contains infinitely many elements in A.

Let $a_{\varepsilon_0} \in A \cap (x_{N_0} - \frac{\varepsilon_0}{2}, x_{N_0} + \frac{\varepsilon_0}{2})$ and $a_{\varepsilon_0} \neq L$.

Then $|x_{N_0} - a_{\varepsilon_0}| < \frac{\varepsilon_0}{2}$. hence $|a_{\varepsilon_0} - L| \leq \underbrace{|a_{\varepsilon_0} - x_{N_0}|}_{\frac{\varepsilon_0}{2}} + \underbrace{|x_{N_0} - L|}_{\frac{\varepsilon_0}{2}} = \varepsilon_0$

$\Rightarrow$ $a_{\varepsilon_0} \in (L-\varepsilon_0, L+\varepsilon_0) \cap A$ and $a_{\varepsilon_0} \neq L$.

D.

Bonus) (20 pts) Prove that there is a sequence (a_n) in $\mathbb{R}$ such that every real number is a cluster point of (a_n).

Since $\mathbb{Q}$ is countable, there is a bijection $f : \mathbb{N} \to \mathbb{Q}$. Let $a_n = f(n)$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $\alpha \in \mathbb{R}$ $\exists$ sequence in $\mathbb{Q}$ converging to α. Hence, there is a subsequence (a_{n_k}) converging to α. Then, any real number is a cluster point of (a_n).

(This is because $\forall \varepsilon > 0$ $(\alpha-\varepsilon, \alpha+\varepsilon)$ contains only many rational numbers.)