2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Any connected subset of $\mathbb{R}^n$ is path connected.

FALSE. Topologists' Sine Curve

2b) Any compact metric space is complete.

TRUE. (Seq. compactness)

2c) Any metric space with discrete metric is complete.

TRUE. (Only Cauchy seq. is constant seq.)

2d) Any Cauchy sequence has a unique cluster point.

FALSE.

may not have any.

$X = (0,1)$

$a_n = \frac{1}{n}$

3) (15 pts) Prove or give a counterexample for the following statement.

Let X be a metric space, and $A \subset X$. A is compact if and only if A is closed and bounded.

FALSE.

$X = [0,1]$

d: discrete metric

$A = X \Rightarrow$ A closed & bdd

but not compact.

A closed: Any subset of discrete space is closed.

A bdd: $A \subseteq B_2(0)$

A is not compact.

Let $U_x = \{x\} \Rightarrow \mathcal{U} = (U_x)_{x \in [0,1]}$ is an open cover

However, there is no finite subcover of $\mathcal{U}$.

4) (15 pts) Prove or disprove the following statement.

$\rho(a,b) = (\sqrt{|a_1 - b_1|} + \sqrt{|a_2 - b_2|})^2$ is a metric on $\mathbb{R}^2$.

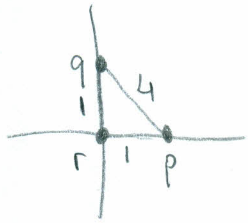
$(a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2)$

NO.

$p = (1,0)$

$q = (0,1)$

$r = (0,0)$



$\rho(p,q) \overset{?}{\leq} \rho(p,r) + \rho(r,q)$

$\rho(p,q) = (\sqrt{|1-0|} + \sqrt{|0-1|})^2 = 2^2 = 4$

$\rho(p,r) = (\sqrt{|1-0|} + \sqrt{0-0}) = 1^2 = 1$

$\rho(r,q) = (\sqrt{0-0} + \sqrt{|0-1|})^2 = 1^2 = 1$

$4 \not\leq 1+1 \Rightarrow$ Triangle Inequality does not hold.

ρ is not a metric on $\mathbb{R}^2$.

5) (15 pts) Let X be a metric space, and $A \subset X$. Show that if $\overline{A}$ is compact, then the derived set A' is also compact.

Any closed subset of a compact metric space is compact.

If we show A' is closed, since $A' \subseteq \bar{A}$, we are done!

Claim: A' is closed.

Proof: $A' = \{ x \in X \mid \forall \varepsilon > 0 \ B_\varepsilon(x) \cap (A - \{x\}) \neq \emptyset \}$

Let $(x_n) \subseteq A'$ and $x_n \to y \in \bar{A}$.

If we show $y \in A'$, we are done.

Let $\varepsilon_0 > 0$ be given

$\exists N$ s.t. $\forall n \geq N$ $d(x_n, y) < \frac{\varepsilon_0}{2}$

Since $x_n \in A'$, $\exists a \in B_{\frac{\varepsilon_0}{2}}(x_n) \cap (A - \{x_n\}) \Rightarrow d(a, x_n) < \frac{\varepsilon_0}{2}$

$d(a,y) \leq \underbrace{d(a,x_n)}_{< \frac{\varepsilon_0}{2}} + \underbrace{d(x_n,y)}_{< \frac{\varepsilon_0}{2}} < \varepsilon_0$ and $a \neq y$

$\Rightarrow y \in A'$. $\Rightarrow A'$ closed $\square$.

6) (25 pts) Check whether the following sets are compact, connected or totally disconnected. Justify your claim.

5a) $\{\cancel{\sin}(1/n) : n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$.

6

Not compact $\frac{1}{n} \to 0$ but $0 \notin A$. Not closed $\Rightarrow$ not compact.

Not connected. $U = (\frac{2}{3}, \infty)$ $V = (0, \frac{2}{3})$ ✓

Totally disconnected. $\forall n$ $\{\frac{1}{n}\} = C_n$ is a component. Show $\{\frac{1}{n}, \frac{1}{m}, \dots\}$ not connected. ✓

5b) $\{(x, e^x) : 0 < x < 1\} \subseteq \mathbb{R}^2$.

6

Not compact. $(\frac{1}{n}, e^{1/n}) \to (0,1) \notin A$ not closed $\Rightarrow$ not compact.

Connected since $(0,1)$ connected: $f : (0,1) \to \mathbb{R}^2$, $x \mapsto (x, e^x)$ cts. ✓

Not totally disconnected.

5c) $\mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$.

7

Not compact. $q_n \to \sqrt{2}$ exist. but $\sqrt{2} \notin A \Rightarrow$ not closed $\Rightarrow$ not compact.

Not connected. $U = (-1, \sqrt{2})$ $V = (\sqrt{2}, 2)$ ✓

Totally disconnected. $\forall q$ $\{q\} = C_q$ is a component. Show $\{q_1, q_2, \dots\}$ not connected.

5d) $\{z : z = x^2 \sin y, x^2 + y^2 \leq 1\} \subseteq \mathbb{R}$.

6

$A = f(D^2)$ where $D^2 = \{(x,y) \mid x^2+y^2 \leq 1\} \subseteq \mathbb{R}^2$ and $f : D^2 \to \mathbb{R}$, $(x,y) \mapsto x^2 \sin y$

D^2 compact and connected $\Rightarrow$ $f(D^2) = A$ compact & connected.

Not totally disconnected.