Math 301
HW 3
Solutions

2- (a) $a_n = 5 \quad \forall n \in \mathbb{N}$. $\forall \varepsilon > 0 \quad a_n \in (5-\varepsilon, 5+\varepsilon) \ \forall n \in \mathbb{N}$
hence 5 is the only cluster point,

(b) $a_n = (-1)^n \ \forall n \in \mathbb{N}$. $\forall \varepsilon > 0 \ a_{2n} \in (1-\varepsilon, 1+\varepsilon) \ \forall n \in \mathbb{N}$ and
$\forall \varepsilon > 0 \ a_{2n-1} \in (-1-\varepsilon, -1+\varepsilon) \ \forall n \in \mathbb{N}$, hence -1 and 1 are the cluster points.

(c) a_n is the remainder when n is divided by 3.
Since $a_{3n} = 0 \ \forall n \in \mathbb{N}$, $a_{3n+1} = 1 \ \forall n \in \mathbb{N}$ and $a_{3n+2} = 2 \ \forall n \in \mathbb{N}$
0, 1 and 2 are cluster points.

(d) $a_n = n \ \forall n \in \mathbb{N}$. (a_n) has no cluster points, as it has no convergent subsequence

(e) $a_n = f(n) \ \forall n \in \mathbb{N}$, $f : \mathbb{N} \longrightarrow \mathbb{Q} \cap [0,1]$ is a bijection.
Let $L \in [0,1]$ and $\varepsilon > 0$. The interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many rational numbers, hence $a_n \in (L-\varepsilon, L+\varepsilon)$ for infinitely many n. Thus any $L \in [0,1]$ is a cluster point of $(a_n)_{n \in \mathbb{N}}$.

8- (i) Since $\mathbb{R}$ is complete every cauchy sequence is convergent.
Consider $a_n = n \ \forall n \in \mathbb{N}$, then (a_n) is not cauchy.

(ii) If $(a_n)_{n \in \mathbb{N}}$ is cauchy in $\mathbb{Z}$, then $a_n = a_m \ \forall n, m \geq N$ for some $N \in \mathbb{N}$, hence (a_n) is convergent,

(iii) Consider $a_n = \sum_{k=0}^{n} \frac{1}{k!}$, then $a_n \in \mathbb{Q} \ \forall n \in \mathbb{N}$. (a_n) is cauchy but not convergent in $\mathbb{Q}$ since $\lim a_n = e \notin \mathbb{Q}$.

10- Given a sequence $(a_n)_{n\in\mathbb{N}}$. let $S=\{a_n : n\in\mathbb{N}\}$.
Let x be an accumulation point of S. Given $\varepsilon>0$.
We show that $a_n\in(x-\varepsilon, x+\varepsilon)$ for infinitely many n.
Assume that there exist finitely many $a_n\in(x-\varepsilon, x+\varepsilon)$.
say $\{a_{n_1},\dots,a_{n_k}\}$. Then let $\varepsilon'=\min\limits_{1\le i\le k}|a_{n_i}-x|$, so
there is no element of S in $(x-\varepsilon', x+\varepsilon')$. contradicts
with x being an accumulation point of S. Thus there
exist infinitely many n s.t. $a_n\in(x-\varepsilon, x+\varepsilon)$.
Hence $\forall\varepsilon>0$ $a_n\in(x-\varepsilon, x+\varepsilon)$ for infinitely many n
and so x is a cluster point of $(a_n)_{n\in\mathbb{N}}$.

12- $(\Rightarrow)$: Suppose a is an accumulation point of S. We construct
a sequence (a_n) in S converging to a s.t. $a_n\neq a_m$ if $n\neq m$.
Let a_1 be any element of S. Put $\alpha=\min\{|a_1-a|, \frac{1}{2}\}$
Since a is an accumulation point $\exists x\in S$ s.t. $|x-a|<\alpha_1$.
Set $a_2=x$ and $\alpha_2=\min\{|a_2-a|, \frac{1}{2^2}\}$. Having constructed
$a_1,\dots,a_{n-1}$. choose $a_n\in S$ s.t. $|a_n-a|<\alpha_{n-1}=\min\{|a_{n-1}-a|, \frac{1}{2^{n-1}}\}$.
Given $\varepsilon>0$. $|a_n-a|<\frac{1}{2^{n-1}}$. Choose $N\in\mathbb{N}$ s.t. $\frac{1}{2^{n-1}}<\varepsilon$ $\forall n\geq N$
then $|a_n-a|<\varepsilon$ $\forall n\geq N$ and $a_n\neq a_m$ if $n\neq m$.
$(\Leftarrow)$: Suppose there is a sequence $(a_n)_{n\in\mathbb{N}}$ in S s.t. $\lim a_n=a$
and $a_n\neq a_m$ if $n\neq m$. Given $\varepsilon>0$. then $\exists N\in\mathbb{N}$: $|a_n-a|<\varepsilon$ $\forall n\geq N$.
Hence $(a-\varepsilon, a+\varepsilon)\cap S\setminus\{a\}\neq\emptyset$ since $a_n\in(a-\varepsilon, a+\varepsilon)$ $\forall n\geq N$.
Thus a is an accumulation point.

17 - (a) $a_n = (-1)^n$ $\quad \limsup a_n = \inf_{n\geq 0} \sup_{k\geq n} a_k$. Since $\sup_{k\geq n} a_k = 1$ for all $n \geq 0$. $\inf_{n\geq 0} \sup_{k\geq n} a_k = 1$. $\quad \liminf a_n = \sup_{n\geq 0} \inf_{k\geq n} a_k$. Since $\inf_{k\geq n} a_k = -1$ for all $n\geq 0$ $\quad \sup_{n\geq 0} \inf_{k\geq n} a_k = -1$.

(b) a_n is the remainder when n is divided by 4.

$\limsup a_n = \inf_{n\geq 0} \sup_{k\geq n} a_k$. As $\sup_{k\geq n} a_k = 3$ for all $n\geq 0$

$\inf_{n\geq 0} \sup_{k\geq n} a_k = 3$. $\liminf a_n = \sup_{n\geq 0} \inf_{k\geq n} a_k$. Since $\inf_{k\geq n} a_k = 0$ $\forall n\geq 0$

$\sup_{n\geq 0} \inf_{k\geq n} a_k = 0$.

(c) $a_n = \frac{1}{n+1}$, Since $\lim a_n = 0$, $\liminf a_n = \limsup a_n = 0$.

(d) $a_n = (-1)^n \frac{2n}{n+1}$ $\quad \limsup a_n = \inf_{n\geq 0} \sup_{k\geq n} a_k$. $\sup_{k\geq n} a_k = 2$ $\forall n\geq 0$ so $\inf_{n\geq 0} \sup_{k\geq n} = 2$, $\liminf a_n = \sup_{n\geq 0} \inf_{k\geq n} a_k$. $\inf_{k\geq n} a_k = -2$ $\forall n\geq 0$ so $\sup_{n\geq 0} \inf_{k\geq n} = -2$.

(e) $a_n = \frac{\sin((n+1)\pi)}{n+1}$ $\quad |a_n| = \left| \frac{\sin((n+1)\pi)}{n+1} \right| \leq \frac{1}{n+1}$

Hence $\lim a_n = 0$ so $\liminf a_n = \limsup a_n = 0$.

(f) $a_n = f(n)$, f is any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0,1]$.

In problem 2-e we have seen that any point in $[0,1]$ is a cluster point of $(a_n)_{n\in\mathbb{N}}$ hence $\limsup a_n = 1$ and $\liminf a_n = 0$.

(g) $a_n = n \quad \forall n \in \mathbb{N}$. $\sup_{k\geq n} a_k = \infty$ so $\inf_{n\geq 0} \sup_{k\geq n} a_k = \infty$, so

$\limsup a_n = \infty$. $\inf_{k\geq n} a_k = a_n$ so $\sup_{n\geq 0} \inf_{k\geq n} = \sup_{n\geq 0} a_n = \infty$.

(h) $a_0 = 0$, $a_{2n+1} = \frac{1}{3} + a_{2n}$, $a_{2n+2} = \frac{1}{3} a_{2n+1}$

If $n = 2k+1$, then $a_n = \sum_{i=1}^{k+1} \frac{1}{3^i}$ and if $\underset{k\geq 1}{n = 2k}$, then $a_n = \sum_{i=2}^{k+1} \frac{1}{3^i}$

Then $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{1}{3} \sum_{i=0}^{n} \frac{1}{3^i} = \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$

and $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{1}{9} \cdot \sum_{i=0}^{n-1} \frac{1}{3^i} = \frac{1}{9} \frac{1}{1-\frac{1}{3}} = \frac{1}{6}$.

Hence (a_n) has two cluster points, and $\limsup a_n = \frac{1}{2}$ and $\liminf a_n = \frac{1}{6}$.

21- Let (a_n) and (b_n) be two bounded sequences.

(a) Given $\varepsilon > 0$. Then by definition there exist finitely many n s.t. $a_n < \liminf a_n - \varepsilon/2$ and $b_n < \liminf b_n - \varepsilon/2$, hence there exists finitely many n s.t. $a_n + b_n < \liminf a_n + \liminf b_n - \varepsilon$. so $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n - \varepsilon$. Since $\varepsilon > 0$ was arbitrary $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$

(b) Given $\varepsilon > 0$. Then by definition there exist finitely many n s.t. $a_n > \limsup a_n + \varepsilon/2$ and $b_n > \limsup b_n + \varepsilon/2$, hence there exists finitely many n s.t. $a_n + b_n > \limsup a_n + \limsup b_n + \varepsilon$. so $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n + \varepsilon$. Since $\varepsilon > 0$ was arbitrary $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.